A LECTURE on SPIN ANGULAR MOMENTUM and PAULI SPIN MATRICES

- MSC 2nd Sem

SPIN ANGULAR MOMENTUM

* The spin is entirely quantum mechanical concept.
* spin is intrinsic quantity. It cannot be measured or defined.
* Unlike the orbital angular momentum, spin can't be defined by the differential operator.

$$\vec{L} = \vec{r} \times \vec{P} \neq \vec{S}$$

General Theory of spin

spin angular momentum follows some commutation relations -

$$[S_x, S_y] = i\hbar S_z \qquad [S^2, S_x] = 0$$
$$[S_y, S_z] = i\hbar S_x \qquad [S^2, S_y] = 0$$
$$[S_z, S_x] = i\hbar S_y \qquad [S^2, S_z] = 0$$

These commutation relations cannot be derived.

$\because$ S^2 commutes with S_z, hence S^2 and S_z can have simultaneous eigen functions with eigen values $s(s+1)\hbar^2$ and $m_s\hbar$ respectively.

Let $|s, m_s\rangle$ be the simultaneous eigen function of S^2 and S_z with eigen value $s(s+1)\hbar^2$ and $m_s\hbar$.

Eigen value Equation for S^2 and S_z -

$$S^2|s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$
$$S_z|s, m_s\rangle = m_s\hbar |s, m_s\rangle$$

Like orbital angular momentum S_+ and S_- can be defined as-

$$S_+ = S_x + iS_y$$
$$S_- = S_x - iS_y$$

Here S_+ is raising operator and S_- is lowering operator

$$S_+|s, m_s\rangle = \frac{\hbar}{2}\sqrt{(s+m)(s+m+1)}\,|s, m_s+1\rangle$$
$$S_-|s, m_s\rangle = \hbar/2\sqrt{(s+m)(s-m+1)}\,|s, m_s-1\rangle$$

Therefore S_x and S_y can be written as-

$$S_x|s, m_s\rangle = \hbar/2\left[\sqrt{(s+m)(s+m+1)}\,|s, m_s+1\rangle + \sqrt{(s+m)(s-m+1)}\,|s, m_s-1\rangle\right]$$

$$S_y |s, m_s\rangle = \hbar/2 \left[\sqrt{(s+m)(s+m+1)}\, |s, m_s+1\rangle - \sqrt{(s+m)(s-m+1)}\, |s, m_s-1\rangle \right]$$

$\because$ we know $S^2 = S_x^2 + S_y^2 + S_z^2$

$\therefore \langle S^2 \rangle = \langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle$

$\langle S^2 \rangle - \langle S_z^2 \rangle = \langle S_x^2 \rangle + \langle S_y^2 \rangle$

and $\langle S_x^2 \rangle = \langle S_y^2 \rangle$

Therefore $\langle S^2 \rangle - \langle S_z^2 \rangle = 2\langle S_x^2 \rangle$

The eigen values of S^2 and S_z are $s(s+1)\hbar^2$ and $m_s\hbar$ respectively.

$$2\langle S_x^2 \rangle = s(s+1)\hbar^2 - m_s^2\hbar^2$$

$$\langle S_x^2 \rangle = \hbar^2/2 \, [s(s+1) - m_s^2]$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{2} [s(s+1) - m_s^2]$$

Therefore, the matrix elements of S^2 and S_z are-

$$\langle s', m_s' | S^2 | s, m_s \rangle = s(s+1)\hbar^2 \, \delta_{s',s} \, \delta_{m_s' m_s}$$

Therefore the matrix form of S^2-

$$S^2 = \begin{array}{c c} & \begin{array}{cc} m_s \rightarrow +\frac{1}{2} & -\frac{1}{2} \end{array} \\ \begin{array}{c} m_s' \\ +\frac{1}{2} \\ \\ -\frac{1}{2} \end{array} & \begin{bmatrix} 3/4\hbar^2 & 0 \\ 0 & -3/4\hbar^2 \end{bmatrix} \end{array}$$

Here eigen value $= s(s+1)$

for $s = 1/2$

$m_s = -\frac{1}{2}$ and $+\frac{1}{2}$ $\quad (\because m_s$ is from $-s$ to $+s)$

Therefore the matrix of S^2 depends on the value of 's' and independent of m_s.

$$\boxed{S^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad \text{--- (1)}$$

The matrix element of S_z -

$$\langle s', m_s' | S_z | s, m_s \rangle = m_s\hbar \, \delta_{s's} \, \delta_{m_s' m_s}$$

Therefore the matrix form of S_z is given by -

Here eigen value is $m_s\hbar$

for $s = \frac{1}{2}$ $\quad m_s = -\frac{1}{2}$ and $+\frac{1}{2}$.

$$S_z = \begin{array}{c|c} & m_s \rightarrow \quad +\frac{1}{2} \qquad\qquad -\frac{1}{2} \\ \begin{array}{c} m_s \downarrow \\ +\frac{1}{2} \\ \\ -\frac{1}{2} \end{array} & \begin{bmatrix} m_s\hbar & 0 \\ 0 & -m_s\hbar \end{bmatrix} \end{array}$$

Here the matrix of S_z independent of 's' but depend on the value of 'm_s'.

$$\boxed{S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \quad \text{— (2)}$$

SPIN - $\frac{1}{2}$ And PAULI SPIN MATRICES

Eigen value Equation of s^2 and s_z can be written as-

$$\left.\begin{aligned} s^2|s, m_s\rangle &= s(s+1)\hbar^2 |s, m_s\rangle \\ \& \; s_z |s, m_s\rangle &= m_s\hbar |s, m_s\rangle \end{aligned}\right\} \quad -(3)$$

$\because \; s^2|s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$

For $s = \frac{1}{2}$; $m_s = \pm\frac{1}{2}$

$$\left.\begin{aligned} |\uparrow\rangle &= s^2|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = 3/4\,\hbar^2 |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \\ |\downarrow\rangle &= s^2|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -3/4\,\hbar^2 |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned}\right] \quad -(4)$$

These are the two possible eigen states of s^2 for $s = \frac{1}{2}$.

Now, $s_z |s, m_s\rangle = m_s\hbar |s, m_s\rangle$

for $s = \frac{1}{2}$; $m_s = \pm\frac{1}{2}$

$$\left.\begin{aligned} |\uparrow\rangle &= s_z|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \hbar/2 \; |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \\ |\downarrow\rangle &= s_z|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\tfrac{\hbar}{2} \; |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned}\right] \quad -(5)$$

These are the two possible eigen states of s_z.

Equations (4) and (5) represents the total wave functions for spin - $\frac{1}{2}$ particles corresponding to spin up and spin down cases.

$|\uparrow\rangle$ and $|\downarrow\rangle$ can be expressed in terms of spinors

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} |\tfrac{1}{2}, +\tfrac{1}{2}\rangle &= |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle &= |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\}$$

PAULI MATRICES

When $s = \frac{1}{2}$, it is convenient to introduce Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ which are related by spin vector.

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

Using the relations of S_x, S_y and S_z we get the following matrix form for Pauli spin matrices of σ_x, σ_y and σ_z.

$$\left.\begin{aligned} \sigma_x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \sigma_z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}\right\} \quad (6)$$

These matrices satisfies following properties

i) $\sigma_j^2 = I$ here $j = x, y, z$

ii) $\sigma_j \sigma_k + \sigma_k \sigma_j = 0$ where $j \neq k$

iii) These Pauli matrices satisfy following commutation relations -

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad — (7)$$

where ϵ_{ijk} is levi-Civita tensor and is defined as-

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ are in cyclic order} \\ 0 & \text{if any two indices are equal} \\ -1 & \text{if } i,j,k \text{ are not in cyclic order} \end{cases}$$

Let $i = x$, $j = y$, $k = z$

then $[\sigma_x, \sigma_y] = 2i \epsilon_{xyz} \sigma_z$

$$\therefore \boxed{[\sigma_x, \sigma_y] = 2i \sigma_z}$$

$\because$ We know

$$[S_x, S_y] = i\hbar S_z$$

and $\vec{S} = \hbar/2 \, \vec{\sigma}$

$$\left[\hbar/2 \, \sigma_x , \frac{\hbar}{2} \sigma_y \right] = i\hbar \left[\frac{\hbar}{2} \sigma_z \right]$$

$$\therefore \boxed{[\sigma_x , \sigma_y] = 2i\sigma_z}$$ {In this way, we can prove the commutation relation}

Now, we have to prove the property -

$$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$$

$$LHS = \sigma_x \sigma_y + \sigma_y \sigma_x$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS$$

Properties -

1) $$\left.\begin{aligned} \sigma_x \sigma_y + \sigma_y \sigma_x &= 0 \\ \sigma_y \sigma_z + \sigma_z \sigma_y &= 0 \\ \sigma_z \sigma_x + \sigma_x \sigma_z &= 0 \end{aligned}\right\} \quad -(1)$$

2) $$\left.\begin{aligned} \sigma_x \sigma_y &= i\sigma_z \\ \sigma_y \sigma_z &= i\sigma_x \\ \sigma_z \cdot \sigma_x &= i\sigma_y \end{aligned}\right\} \quad -(2)$$

3) $$\left.\begin{aligned} [\sigma_x , \sigma_y] &= 2i\sigma_z \\ [\sigma_y , \sigma_z] &= 2i\sigma_x \\ [\sigma_z , \sigma_x] &= 2i\sigma_y \end{aligned}\right\} \quad -(3)$$

4) $$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{ij} \quad -(4)$$

Some properties of Pauli spin operators

1) Pauli spin operator is Hermitian.

$$\sigma_j^{*} = \sigma_j$$

Since spin is an observable therefore, it must possess real eigen value and for this the choosen operator is Hermitian operator.

Pauli spin operators are traceless.

$Tr(\sigma_j) = 0$

3) Eigen values of pauli spin operators are ± 1.

$S_z = \frac{\hbar}{2}\sigma_z$ and $-\frac{\hbar}{2}\sigma_z$

$S_z|s, m_s\rangle = \frac{\hbar}{2}|s, m_s\rangle$

$\frac{\hbar}{2}\sigma_z|s\ m_s\rangle = \frac{\hbar}{2}|s, m_s\rangle$

$\sigma_z|s\ m_s\rangle = +1\,|s, m_s\rangle \quad \text{—(i)}$

This is an eigen value eq of σ_z with eigen value +1.

$S_z|s, m_s\rangle = -\frac{\hbar}{2}|s, m_s\rangle$

$\frac{\hbar}{2}\sigma_z|s, m_s\rangle = -\frac{\hbar}{2}|s, m_s\rangle$

$\sigma_z|s, m_s\rangle = -1\,|s, m_s\rangle \quad \text{—(ii)}$

This is an eigen value eq of σ_z with eigen value -1.

$\therefore$ Eigen values are ± 1.

4.) $Det(\sigma_j) = -1$

5.) $\sigma_x\sigma_y\sigma_z = i$

taking LHS $= \sigma_x\sigma_y\sigma_z$

$= (\sigma_x\sigma_y)\sigma_z$

$= (i\sigma_z)\sigma_z$ from eq (2)

$= i\sigma_z^2$ (from the prop. $\sigma_j^2 = I$)

$= i =$ RHS

$\therefore \sigma_x\sigma_y\sigma_z = i$

6) For any two operator commutes with $\vec{\sigma}$, we have

$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (A\cdot B)I + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

7) From commutation relation

$e^{i\alpha\sigma_j} = I\cos\alpha + i\sigma_j \sin\alpha$

Bandana Jadawn