SCHOOL OF STUDIES IN PHYSICS JIWAJI UNIVERSITY GWALIOR

Topic: **Magic Numbers and Shell Model**

Magic Numbers

It has been found that the nuclei with proton number or neutron number equal to certain numbers **2,8,20,28,50,82 and 126** behave in a different manner when compared to other nuclei having neighboring values of Z or N. Hence these numbers are known as magic numbers. This is found to be in accordance with the observed nature of elements with filled shells. Thus Physicists looked at such a possibility in case of filling of nucleons in the nucleus. Thus a new model of nucleus has emerged. This model is known as the Shell model.

Experimental evidences for the existence of magic numbers

- The binding energy of magic numbered nuclei is much larger than the neighboring nuclei. Thus larger energy is required to separate a single nucleon from such nuclei.
- Number of stable nuclei with a given value of Z and N corresponding to the magic number are much larger than the number of stable nuclei with neighboring values of Z and N. For example, Sn with Z=50 has 10 stable isotopes, Ca with Z=20 has six stable isotopes.
- Naturally occurring isotopes whose nuclei contain magic numbered Z or N have greater relative abundances. For example, Sr-88 with N=50, Ba-138 with N=82 and Ce-140 with N=82 have relative abundances of 82.56%, 71.66% and 88.48% respectively.
- Three naturally occurring radioactive series decay to the stable end product Pb with Z=82 in three isotopic forms having N=126 for one of them.
- Neutron absorbing cross section is very low for the nuclei having magic numbered neutron number.
- Nuclei with the value of N just one more than the magic number spontaneously emit a neutron (when excited by preceding beta-decay) E.g., O-17, K-87 and Xe-137.
- Nuclei with magic numbers of neutrons or protons have their first excited states at higher energies than in cases of the neighboring nuclei.
- Electric quadrupole moment of magic numbered nuclei is zero indicating the spherical symmetry of nucleus for closed shells.

- Energy of alpha or beta particles emitted by magic numbered radioactive nuclei is larger than that from other nuclei.

Double Magic Numbers: When both proton number (Z) and neutron number (N) are magic numbers e.g $_{2}He_{2}^{4}$, $_{8}O_{8}^{16}$, $_{20}Ca_{20}^{40}$, $_{82}Pb_{126}^{208}$. They show exceptionally high stability.

Singly Magic Numbers: When either N or Z are among magic numbers e.g., $_{38}Sr_{50}^{88}$

Semi-magic numbers: Besides the above mentioned magic numbers there are other numbers of Z or N at which B.E/A is high but in the other properties the evidences for special stability of nuclei are less marked than that of magic number. These numbers are known as semi-magic numbers and include **14, 28, and 40.**

Shell Model

These magic numbers can be explained in terms of the Shell Model of the nucleus, which considers each nucleon moves in an orbit within nucleus independent of other nucleons. The orbit is determined by potential energy function *V(r),* which represents the average of all interactions with other nucleons and is same for each particle. The model is similar to the shell model of atom except to the force term. The potential term *V(r)* is analogous to Coulomb potential term in atomic shell model. Also the orbit of nucleons is analogous to an orbit state of electron.

As the nucleons are arranged in shells inside nucleus, each shell can allow limited number of nucleons. When a shell is completely filled the resulting configuration is particularly stable and has usually low energy.

For a spherically symmetric potential the wavefunction (neglecting its spin for the moment) for any nucleon whose coordinates from the centre of the nucleus are given by polar coordinates is of the form $\psi_{nlm} = R_{nl}(r)Y_{l}^{m}(\theta,\phi)$

The energy eigenvalues will depend on the principle quantum number '*n*' and the orbital angular momentum '*l*' but are degenerate in the magnetic quantum number '*m*'. These energy levels come in 'bunches' called "shells" with a large energy gap just above each shell. In their ground state the nucleons fill up the available energy levels from the bottom upwards with two protons (neutrons) in each available proton (neutron) energy level. The properties of this potential are completely unknown - so we need to take a guess. We begin by considering the infinite well potential and the harmonic oscillator potential. The eigen states available to nucleons of mass

'M' moving in spherical symmetric potential are determined by solving Schrödinger Wave Equation

$$\left\{\nabla^2 + \frac{2M}{\hbar^2}\left[E - V(r)\right]\right\}\psi(r) = 0$$

Solution of this equation can be of the form

$$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\phi)$$

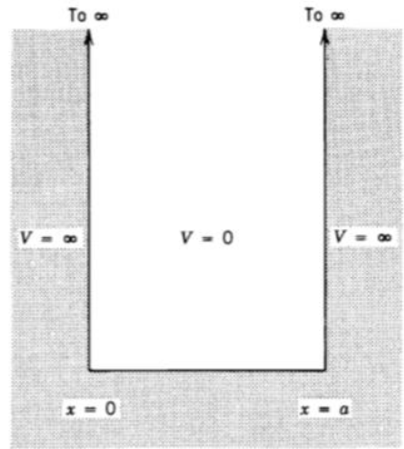

Fig. 1: 1-D Infinite potential Well

Infinite Potential Well

The infinite potential well is shown in the Fig. 1 and mathematical form potential is given by;

$$V(r) = \begin{cases} 0 & \text{for } 0 \le r \le a \\ \infty & \text{otherwise} \end{cases}$$

Clearly, the wavefunction ψ is only non-zero in the region $0 \le r \le a$. Within this region, it is subject to the physical boundary conditions that it be well behaved (*i.e.*, square-integrable) at *r* = *0*, and that it be zero at *r* = *a*. Writing the wavefunction in the standard form

$$\psi(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

The radial function $R_{n,l}$ satisfies

$$\frac{d^2R_{n,l}}{dr^2} + \frac{2}{r}\frac{dR_{n,l}}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R_{n,l} = 0$$

In the region $0 \le r \le a$, where

$$k^2 = \frac{2mE}{\hbar^2}.$$

Defining the scaled radial variable $z = kr$, the above differential equation can be transformed into the standard form

$$\frac{d^2R_{n,l}}{dz^2} + \frac{2}{z}\frac{dR_{n,l}}{dz} + \left[1 - \frac{l(l+1)}{z^2}\right]R_{n,l} = 0.$$

Solution of the equation is given by spherical Bessel function

$$j_l(z) = z^l\left(-\frac{1}{z}\frac{d}{dz}\right)^l\left(\frac{\sin z}{z}\right),$$

The energy eigen values are determined by bondary condition $j_l(z) = 0$; given by

$$E_{nl} = z_{nl}^2 \frac{\hbar^2}{2ma^2}$$; where z_{nl} is the n^th zero of $j_l(z)$

The l^{th} eigen state is $2(2l+1)$ degenerate, $(2l+1)$ due due to m_l degeneracy and 2 due to m_s degeneracy. The energy level diagram is shown in Fig. 2.

Harmonic Oscillator Potential

If the potental has form $V(r) = M\omega^2 r^2/2$, then by solving Schrödinger Wave Equation we will have;

$E = (2n + l + 3/2)\hbar\omega$; Where $n = 0,1,2\ldots\ldots$; $l = 0,1,2\ldots\ldots$

Put $N = 2n+l$; $E = (N + 3/2)\hbar\omega$

For $N = 0$: n 0; $l = 0$

For $N = 1$: $n = 0$; $l = 1$

For $N = 2$: $n = 1$; $l = 0$ and $n = 0$; $l = 2$ and so on….

Therefore energy level diagram and capacity of each level is shown in Fig. 2.

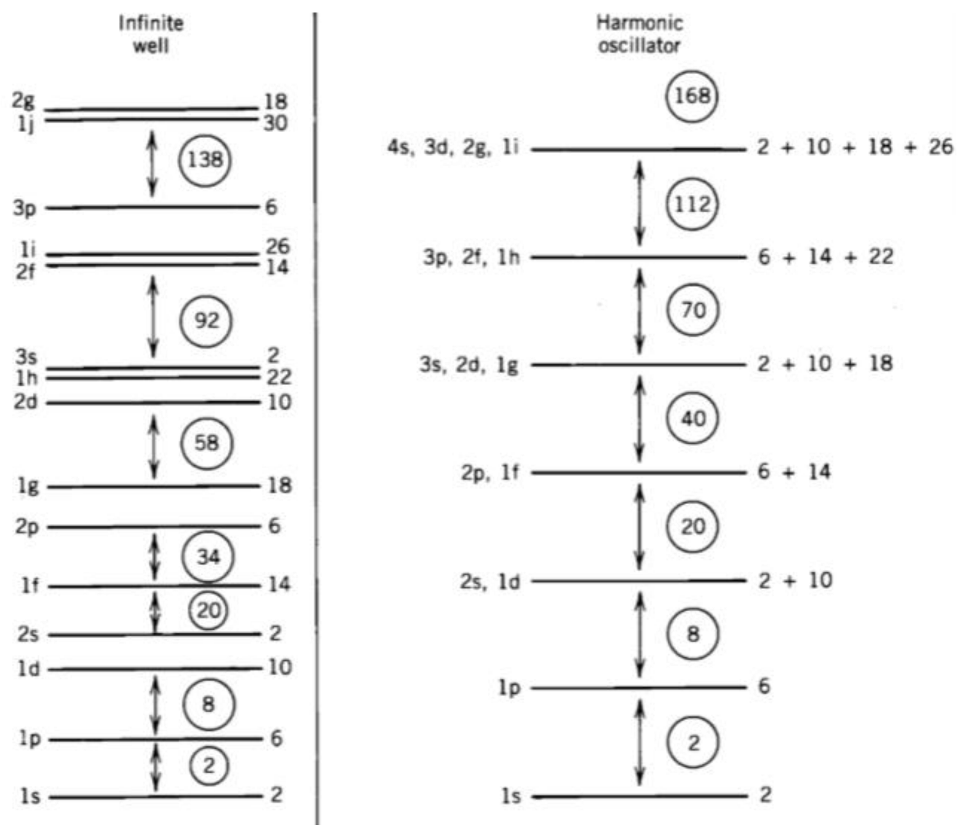

Fig. 2: Shell structure obtained with infinite well and harmonic oscillator potentials. The capacity of each level is indicated to its right. Large gaps occur between the levels, which we associate with closed shells. The circled numbers indicate the total number of nucleons at each shell closure

Note: As in atomic physics, we use spectroscopic notation to label the levels, with one important exception: the index n is not the principal quantum number, but simply counts the number of levels with that l value. Thus *1d* means the first (lowest) *d* state, *2d* means the second, and so on. (In atomic spectroscopic notation, there are no *1d* or *2d* states).

From Fig. 2 it is evident that

magic numbers of 2, 8, and 20 emerging in both of these schemes, but the higher levels do not correspond at all to the observed magic numbers.

As a first step in improving the model, we try to choose a more realistic potential. The infinite well is not a good approximation to the nuclear potential for several reasons: To separate a neutron or proton, we must supply enough energy to take it out of the well- an infinite amount! In addition, the nuclear potential does not have a sharp edge, but rather closely approximates the nuclear charge and matter distribution, falling smoothly to zero beyond the mean radius R. The harmonic oscillator, on the other hand, does not have a sharp enough edge, and it also requires infinite separation energies. Instead, we choose an intermediate form : $V(r) = \frac{-V0}{1+\exp[(r-R)/a]}$;which is sketched in Fig. 3.

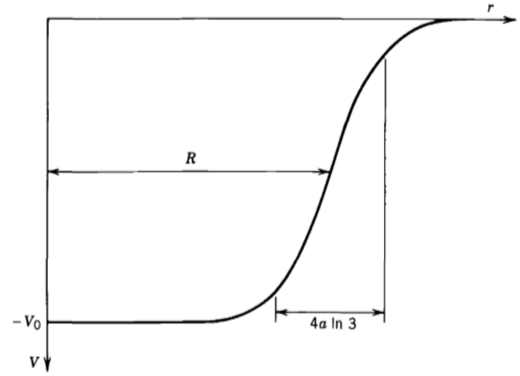

Fig. 3: A realistic form for the shell-model potential. The "skin thickness" 4a In 3 is the distance over which the potential changes from 0.94 to 0.16.

The parameters *R* and *a* give, respectively, the mean radius and skin thickness, R = 1.25A1/3 fm and a = 0.524 fm. The well depth V_0 is adjusted to give the proper separation energies and is of order 50 MeV. The resulting energy levels are shown in Fig. 4. The effect of the potential, as compared with the harmonic oscillator (Fig. 2) is to remove the *l* degeneracies of the major shells. As we go higher in energy, the splitting becomes more and more severe, eventually becoming as large as the spacing between the oscillator levels themselves. Filling the shells in order with 2(2*l*+ 1) nucleons, we again get the magic numbers 2, 8, 20 and 40 but the higher magic numbers do not emerge from the calculations. To get correct magic numbers Spin-Orbit Interaction should be accounted.

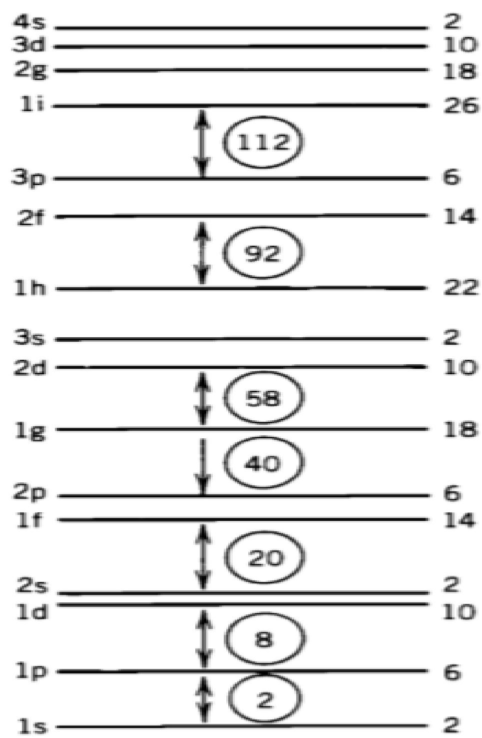

Fig. 4: Shell structure obtained with infinite well and harmonic oscillator potentials