2. Radiation and Retarded Potentials

2.2 Liénard-Wiechert Potentials and Point Charges

Retarded Potentials and the Wave Equation

We have arrived at a modified form of the vector and scalar potentials in terms of a charge density and current density source terms evaluated at a retarded time. Before proceeding we are required to verify that these vector and scalar potentials discussed in 2.1, proposed on physical grounds, do indeed satisfy the inhomogeneous wave equations.

Alfred-Marie **Liénard, French Physicist** (1869-1958)

Emil Johann **Wiechert, German Physicist** (1861-1928)

Equations were developed in part by Alfred-Marie Liénard in 1898 and independently by Emil Wiechert in 1900 and continued into the early 1900s

Here we will verify that retarded potentials V and **A:**

$$V(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\vec{r}\,',t-\frac{1}{c}R\right)}{R}d^3r',$$

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\left(\vec{r}\,',t-\frac{1}{c}R\right)}{R}d^3r',$$

$$\text{where } R=\left|\vec{r}-\vec{r}\,'\right|,\ \vec{B}=\nabla x\vec{A},\ \ \vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t},$$

satisfy the inhomogeneous wave equation for V:

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V=-\frac{\rho}{\varepsilon_0}$$

$$\text{Firstly, consider } \nabla^2_{\vec{r}}=\nabla^2_{\vec{R}} \text{ where } \vec{R}=\vec{r}-\vec{r}\,'$$

$$\nabla^2_{\vec{r}}V=\frac{1}{4\pi\varepsilon_0}\int_V d^3r\,\nabla^2_{\vec{r}}\frac{\rho(R)}{R} \quad \leftarrow \text{No } \theta,\ \phi \text{ dependence in integration}$$

$$\text{Now: } \nabla^2_{\vec{R}}\frac{\rho}{R}=\frac{1}{R}\frac{\partial^2\rho}{\partial^2 R}+\rho\nabla^2_{\vec{R}}\frac{1}{R}$$

$$\text{Recall: } \frac{1}{4\pi}\nabla^2_{\vec{R}}\left(\frac{1}{R}\right)=-\delta^3(\vec{R})$$

As an aside, remember for a point charge at the origin: $V=\frac{q}{4\pi\varepsilon}\frac{1}{r}$ and $\nabla^2 V=-\rho/\varepsilon_0$

.

$$\nabla^2\frac{q}{4\pi\varepsilon_0}\frac{1}{r}=-\frac{q}{\varepsilon_0}\delta^3(r) \ \Rightarrow \frac{1}{4\pi}\nabla^2\frac{1}{r}=-\delta^3(r) \text{ or for a charge at } \vec{r}\,': \ \boxed{\frac{1}{4\pi}\nabla^2\frac{1}{R}=-\delta^3(R),\ R=\left|\vec{r}-\vec{r}\,'\right|}$$

Also, $\rho = \rho(t - R/c)$ satisfies the 1-D wave equation:

$$\frac{\partial^2 \rho}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = 0 \;\rightarrow \frac{\partial^2 \rho}{\partial R^2} = \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2}$$

Thus, $\nabla^2_{\vec{R}} V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V d^3r' \left[\frac{1}{R}\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} - 4\pi\rho\delta^3(\vec{r} - \vec{r}\,')\right]$

$$\nabla^2_{\vec{R}} V = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\underbrace{\frac{1}{4\pi\varepsilon_0}\int_V d^3r' \frac{\rho(r', t - R/c)}{R}}_{V(r, t-R/c)}\right] - \frac{\rho(r, t - R/c)}{\varepsilon_0}$$

$$\Rightarrow \nabla^2_{\vec{R}} V = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} V - \frac{\rho(\vec{r}, t - R/c)}{\varepsilon_0} \text{ or } \boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V\left(r, t - R/c\right) = -\frac{\rho(r, t - R/c)}{\varepsilon_0}}$$

Thus, the proposed representation in terms of retarded potentials do indeed satisfy the wave equation (it is worth noting that a similar representation for retarded **E** and **B** fields in terms of t-R/c do *not* satisfy the wave equation).

Finally, we note that the advanced potentials with t = t +R/c are entirely consistent with Maxwell's equations but they violate **causality** –in nature effect does not precede cause! For completeness we include the advanced potentials:

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\vec{r}\,', t + \frac{1}{c}R\right)}{R} d^3r',$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\left(\vec{r}\,', t + \frac{1}{c}R\right)}{R} d^3r'.$$

They do have some theoretical interest as they are a product of the time-reversal invariance of the wave equations.

Exercise:

Show that $\nabla^2_{\vec{R}} \dfrac{\rho}{R} = \dfrac{1}{R}\dfrac{\partial^2 \rho}{\partial R^2} + \rho\nabla^2_{\vec{R}} \dfrac{1}{R}$

(Hint use spherical coordinates for $\nabla^2_{\vec{R}}$ and $\nabla_{\vec{R}}$)

ELECTRODYNAMICS: PHYS 30441

Liénard-Wiechert Potentials and Point Charges

We will investigate the scalar and vector potential generated by a moving point charge q.

$$\rho(\vec{r}\,',t') = q\delta^3\left[\vec{r}\,'-\vec{r}_0^{\,'}(t')\right], \quad t' = t - \frac{1}{c}\left|\vec{r}-\vec{r}\,'\right| \quad \rightarrow \text{Integration is difficult as it also depends on } \vec{r}$$

$$\rho(\vec{r}\,',t') = q\int d\tau\,\underbrace{\delta\left(\tau - t + \frac{1}{c}\left|\vec{r}-\vec{r}\,'\right|\right)}_{\text{contribution only at } \tau=t'=t-\frac{1}{c}\left|\vec{r}-\vec{r}\,'\right|}\;\delta^3\underbrace{\left(\vec{r}\,'-\vec{r}_0^{\,'}(\tau)\right)}_{\tau \text{ and } \vec{r} \text{ are no longer connected}}$$

$$\Rightarrow V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int d\tau\,\frac{\delta\left[\tau - t + \frac{1}{c}R(\tau)\right]}{\underbrace{\left|\vec{r}-\vec{r}_0^{\,'}(\tau)\right|}_{R(\tau)}}$$

Now we use the sifting property of delta functions:

$$\delta[f(x)] = \sum_{f(x_0)=0}\frac{\delta(x-x_0)}{\left|\frac{d}{dx}f(x)\right|}, \text{ e.g. } \delta(ax) = \frac{1}{|a|}\delta(x)$$

$$f(\tau) = \tau - t + \frac{1}{c}R(\tau), \quad \frac{d}{d\tau}f = 1 + \frac{1}{c}\frac{dR}{d\tau}\bigg|_{\tau=t_0^{'}} \quad (\text{here } t_0^{'} = t_{ret})$$

$$\Rightarrow V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R(t_0^{'})}\frac{1}{\left[1+\frac{1}{c}\frac{dR(\tau)}{d\tau}\Big|_{\tau=t_0^{'}}\right]}$$

Exercise:

$$\text{Prove } \frac{1}{c}\frac{dR(\tau)}{d\tau}\bigg|_{\tau=t_{ret}} = -\frac{\vec{\beta}.\vec{R}}{R}\bigg|_{\tau=t_{ret}} \quad \left(\text{where } c\vec{\beta} = \frac{d\vec{r}_0^{\,'}}{d\tau} \text{ -particle's velocity at time } \tau\right)$$

Ans:

$$\text{Consider } \frac{1}{c}\frac{dR}{d\tau}\bigg|_{\tau=t_{ret}} = \frac{1}{c}\left(\nabla_{\vec{r}_0^{\,'}}\left|\vec{r}-\vec{r}_0^{\,'}\right|\right).\frac{d\vec{r}_0^{\,'}}{d\tau}\bigg|_{\tau=t_{ret}}$$

$$\frac{1}{c}\frac{dR}{d\tau}\bigg|_{\tau=t_{ret}} = -\frac{\left(\vec{r}-\vec{r}_0^{\,'}\right).\vec{\beta}}{\left|\vec{r}-\vec{r}_0^{\,'}\right|}\bigg|_{\tau=t_{ret}} = -\frac{\vec{R}.\vec{\beta}}{R}\bigg|_{\tau=t_{ret}}$$

Using the result from the exercise

$$\Rightarrow \boxed{V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(R-\vec{\beta}.\vec{R}\right)_{ret}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[R\left(1-\vec{\beta}.\hat{R}\right)\right]_{ret}}}$$

where the quantities in parenthesis are evaluated at the retarded time: $t_{ret} = t - \frac{1}{c}R(t_{ret})$.

Also, for the vector potential:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V d^3r'\frac{\vec{J}(\vec{r}',t')}{\left|\vec{r}-\vec{r}'\right|}, \text{ with } \vec{J}(\vec{r}',t') = c\vec{\beta}(t')\rho(\vec{r}',t') = qc\vec{\beta}(t')\delta^3[\vec{r}-\vec{r}_0'(t')] \text{ and } t'=t-\frac{1}{c}\left|\vec{r}-\vec{r}'\right|$$

and this gives the <u>Liénard-Wiechert vector potential</u>:

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc[\beta]_{ret}}{\left[R-\vec{\beta}.\vec{R}\right]_{ret}} = \frac{\mu_0}{4\pi}\frac{qc[\beta]_{ret}}{\left[R\left(1-\vec{\beta}.\hat{R}\right)\right]_{ret}}}$$

For $\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B}=\vec{\nabla}x\vec{A}$ we can derive the Lienard-Wiechert fields:

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{\left(\hat{R}-\vec{\beta}\right)\left(1-\beta^2\right)}{\left(1-\vec{\beta}.\hat{R}\right)^3R^2}+\frac{\hat{R}x\left(\left(\hat{R}-\vec{\beta}\right)x\dot{\vec{\beta}}\right)}{c\left(1-\vec{\beta}.\hat{R}\right)^3R}\right]_{ret} \;;\; \dot{\vec{\beta}}=\frac{d}{d\tau}\vec{\beta}(\tau),$$

$$\vec{B}=\frac{\mu_0 qc}{4\pi}\left[\frac{\left(\vec{\beta}x\hat{R}\right)\left(1-\beta^2\right)}{\left(1-\vec{\beta}.\hat{R}\right)^3R^2}+\frac{\dot{\vec{\beta}}.\hat{R}(\vec{\beta}x\hat{R})}{c\left(1-\vec{\beta}.\hat{R}\right)^3R}+\frac{\dot{\vec{\beta}}x\hat{R}}{c(1-\vec{\beta}.\hat{R})^2R}\right]_{ret}$$

For slow moving charges these reduce to the results of electro and magneto-statics. To prove this consider $\vec{\beta},\dot{\vec{\beta}} << 1$.

Also, it is straightforward to show $\vec{B} = \frac{1}{c}\left[\hat{R}\right]_{ret}\times\vec{E}$ (as an exercise show this is indeed true)

Features of Liénard-Wiechert Potentials

- The Lienard-Wiechert potentials and fields are relativistically correct (Lorentz covariance)
- For point charges, the B-field is always perpendicular to the E-field and the unit vector $\hat{R}$ from the retarded position to the observation point
- We have already seen that the **E** and **B** fields can be decomposed into velocity and acceleration components: $\vec{E} = \vec{E}_v + \vec{E}_a$, $\vec{B} = \vec{B}_v + \vec{B}_a$

  with $\vec{E}_v, \vec{B}_v \propto \vec{\beta}, \frac{1}{R^2}$ and $\vec{E}_a, \vec{B}_a \propto \dot{\vec{\beta}}, \frac{1}{R}$

- The energy radiated by a moving charge per unit time per unit solid angle is:

$$\boxed{\frac{d\overline{P}}{d\Omega} = \lim_{R->\infty}\left\{\left(\vec{P}.\hat{R}R^2\right).\frac{dt}{dt_{ret}}\right\}}$$ at large distances

$\vec{P} = \vec{E}x\vec{H}$ is the Poynting vector which repsents the energy flow per unit area $\perp$ ar to the direction of energy flow.

The components of the Poynting vector are:

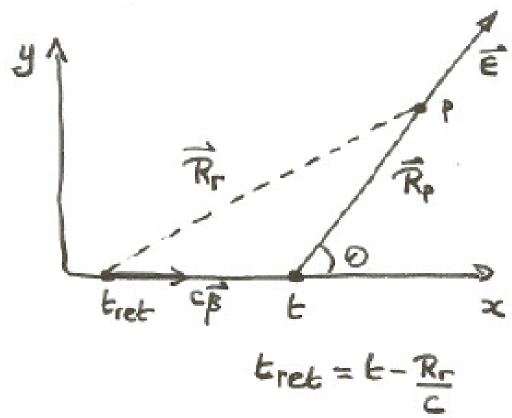
$$P_{vv} \sim \left|\vec{E}_v x \vec{B}_v\right| \propto \frac{1}{R^4},\ P_{va} \sim \left|\vec{E}_v x \vec{B}_a\right| \propto \frac{1}{R^3}$$

$$P_{aa} \sim \left|\vec{E}_a x \vec{B}_a\right| \propto \frac{1}{R^2}$$ $\leftarrow$ the only non-vanishing component

Radiation by an Accelerated Charged Particle

Particle Moving with a Constant Velocity (= cβ)

In this case only the velocity component of the Lienard-Wiechert field contributes:



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\left(\hat{R}-\vec{\beta}\right)\left(1-\beta^2\right)}{\left(1-\vec{\beta}.\hat{R}\right)^3 R^2}\right]_{ret}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\left(1-\beta^2\right)}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^2}\left(\hat{R}_r-\vec{\beta}\right)$$

with $\vec{R}_r = \vec{R}_p + c\vec{\beta}(t - t_{ret})$

or equivalently, $\vec{R}_p = \vec{R}_r - c\vec{\beta}(t - t_{ret})$; also $R_r = (t - t_{ret})c$

$$\Rightarrow \vec{R}_p = \vec{R}_r - \vec{\beta}R_r \text{ or } \vec{R}_p = R_r(\hat{R}_r - \vec{\beta}) \tag{1}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\left(1-\beta^2\right)}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^3}\vec{R}_p}$$

Thus despite the retarded field analysis the field is directed from the actual position of the particle ($\vec{R}_p$) at time t rather than the retarded position!

The objective is now to obtain the denominator in terms of R_p and in terms of the angle between the direction of travel and the field observation point.

$$sq(1) \Rightarrow R_p^2 = R_r^2 - 2R_r\vec{\beta}.\vec{R}_r + R_r^2\beta^2 \tag{2}$$

$$\text{sq denominator} \Rightarrow R_r^2(1-\beta\hat{R}_r)^2 = R_r^2 - 2R_r\vec{\beta}.\vec{R}_r + (\vec{\beta}.\vec{R}_r)^2 \tag{3}$$

Also, perpendicular components are equal:

$$\left|\vec{R}_r \times \vec{\beta}\right|^2 = \left|\vec{R}_p \times \vec{\beta}\right|^2$$

$$R_r^2\beta^2 - (\vec{R}_r.\vec{\beta})^2 = R_p^2\beta^2 - (\vec{R}_p.\vec{\beta})^2 \tag{4}$$

And from (2): $R_r^2 = R_p^2 + 2R_r\vec{\beta}.\vec{R}_r - R_r^2\beta^2$, substitute into (3)

$$R_r^2(1-\vec{\beta}.\hat{R}_r)^2 = R_p^2 + 2R_r\vec{\beta}.\vec{R}_r - R_r^2\beta^2 - 2R_r\vec{\beta}.\vec{R}_r + (\vec{R}_r.\vec{\beta})^2$$

and from (4): $(\vec{R}_r.\vec{\beta})^2 - R_r^2\beta^2 = (\vec{R}_p.\vec{\beta})^2 - R_p^2\beta^2$

$$\Rightarrow R_r^2(1-\vec{\beta}.\hat{R}_r)^2 = R_p^2 + (\vec{R}_p.\vec{\beta})^2 - R_p^2\beta^2 = R_p^2 + \beta^2 R_p^2(\cos^2\theta - 1) = R_p^2(1-\beta^2\sin^2\theta)$$

$$\Rightarrow R_r^2(1-\vec{\beta}.\hat{R}_r)^2 = R_p^2(1-\beta^2\sin^2\theta)$$

i.e. this allows the field to be written in terms of the present location of the particle.

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)\hat{R}_p}{R_p^2(1-\beta^2\sin^2\theta)^{3/2}}}$$

Also, since the Lienard-Wiechert equations give:
$\vec{B} = \frac{1}{c}\hat{R}_r \times \vec{E}$ (where $\hat{R}_r$ is evaluated at the retarted position).
In terms of the present position of the particle $\vec{R}_r = c\vec{\beta}(t - t_{ret}) + \vec{R}_p$,

and $\hat{R}_r = \vec{R}_r / R_r = c\vec{\beta}(t - t_{ret}) / R_r + \vec{R}_p / R_r$; and $R_r = c(t - t_{ret})$

$$\Rightarrow \hat{R}_r = \vec{\beta} + \vec{R}_p / R_r \text{ and } \vec{B} = \frac{1}{c}\left(\vec{\beta} + \vec{R}_p / R_r\right) x\vec{E}$$

and as $\vec{R}_p$ lies along the field then the second term is zero:

$$\Rightarrow \boxed{\vec{B} = \frac{1}{c}\vec{\beta} x\vec{E}}$$

This allows the magnetic field to be obtained:

$$\boxed{\vec{B} = \frac{q}{4\pi\varepsilon_0 c}\frac{(1-\beta^2)\left(\vec{\beta} x\hat{R}_p\right)}{R_p^2(1-\beta^2\sin^2\theta)^{3/2}}} \text{ also note, } \frac{q}{4\pi\varepsilon_0 c} \equiv \frac{q\mu_0 c}{4\pi}$$

Again, the magnetic field does not depend on the position in retarded time but depends on the present position of the particle only!

Features of Particle Moving with Constant Velocity

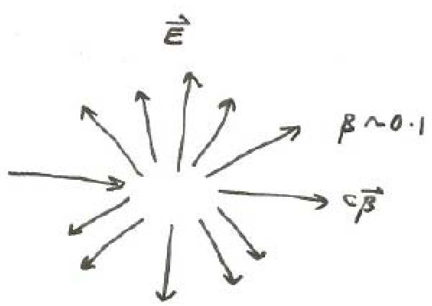
- For relativisitic velocities , i.e. $\beta \sim 1$:

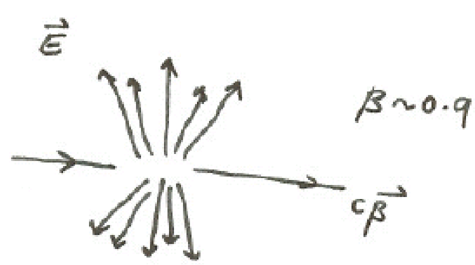
1. At $\theta = \pi/2$ the E-field is enhanced by a factor of $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

2. At $\theta = 0, \pi$ the E-field is suppressed by a factor of $1-\beta^2 = \frac{1}{\gamma^2}$

The consequence of this is the radiation in linear accelerator (applicable for example, to the 2-mile linac in California known as SLAC) is markedly transverse to the direction of motion. For ultra-relavisitic electron beams the radiation is suppressed rapidly by $1/\gamma^2$ in the direction of motion and is enhanced by γ transverse to the motion. Thus, the radiation E-field is concentrated in a pancake-like distribution. Indeed, as $v \to c$ the angle of the pancake goes to $\pi/2$ and the E-field is then entirely transverse to the direction of motion.

- The Poynting vector $\vec{P} = \frac{1}{\mu_0}\vec{E}x\vec{B} \propto \frac{1}{R_p^4}$ and represents an electromagnetic field energy flow attached and convected along with the charge. Also, due to $\lim_{R_p \to \infty}\left\{R_p^2\left|\vec{P}\right|\right\} = 0$, there is <u>no radiation from a uniformly moving charge</u>





Low-velocity beam (β~.1)   Relativistic beam (β~.1)