

**PT-204: Numerical computation method**

UNIT IV

**Numerical Solution of  
Ordinary Differential Equations**

or

**First Order and First Degree Differential Equations**

Euler's Method  
and  
Picard's Method

## (1) Ordinary differential equations

Consider  $y(x)$  to be a function of a variable  $x$ . An **Ordinary differential equation** is an equation relating  $y$ ,  $x$ , first-order derivative of  $y$ . The most general form is :

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

The variable  $y$  is known as a dependent variable, and  $x$  is the independent variable. The equation is of first-order as the highest derivative of  $y$  present in the equation is one.

$Y = g(x)$  is a **solution** of the first-order differential equation (1) means

- i)  $y(x)$  is differentiable
- ii) Substitution of  $y(x)$  and  $y'(x)$  in (1) satisfies the differential equation identically

Mathematical representations of many real-world problems are, commonly, modeled in the form of differential equations. Finding a solution to the differential equation is then critical to that real-world problem.

We expressed the decay of radioactive material as

$$-\frac{dN}{dt} \propto N$$

Or

$$\frac{dN}{dt} = -\lambda N \quad (2)$$

Where  $N$  represents the amount of material at any given time, and  $\lambda$  is a constant.

It may be noted that

$$N = C \exp(-\lambda t) \quad (3)$$

is the solution of differential equations as it identically satisfies the given differential equation for arbitrarily chosen constant  $C$ . This means that the differential equation has infinitely many solutions for different choices of  $C$ . In other words, the real world problem has **infinitely many solutions** which we know is not true.

In fact, an initial condition should be specified for finding the unique solution of the problem. That is, the amount of radioactive material present at time  $t=0$  is  $N_0$ .

When this initial condition is imposed on the solution, the constant  $C$  is evaluated as  $N_0$  and the solution

$$N = N_0 \exp(-\lambda t) \quad (4)$$

is now unique. The expression can now be used for computing the amount of material at any given time.

The solution with arbitrary constant is known as the **general solution** of the differential equation. The solution obtained using **the initial condition** is a **particular solution**.

A first order differential equation together with specified initial condition at  $t=t_0$  is known as first order **Initial Value Problem** (IVP). Mathematically it is defined as

$$y' = f(x, y); \quad x_0 \leq x \leq b \quad \text{with } y(x_0) = y_0 \quad (5)$$

There exist several methods for finding solutions of differential equations. However, all differential equations are not solvable. In other words we can write that the vast majority of first order differential equations can't be solved.

Those differential equations which are either not solvable or it is very difficult to obtain their solution, in such cases, the approximate solution of given differential equation can be obtained.

## (2) **Approximate Solution**

The classical methods for approximate solution of an IVP are:

- i) Euler's method
- ii) Picard Iteration method
- iii) Taylor Series method

### ***2.1 Eulers method***

In this section we'll take a brief look at a fairly simple method for approximating solutions to differential equations. We derive the formulas used by Euler's Method and give a brief discussion of the errors in the approximations of the solutions.

Well, we are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called, oddly enough, Euler's Method. Let's start with a general first order IVP

$$y' = f(x, y); \quad x_0 \leq x \leq b \quad \text{with } y(x_0) = y_0 \quad (6)$$

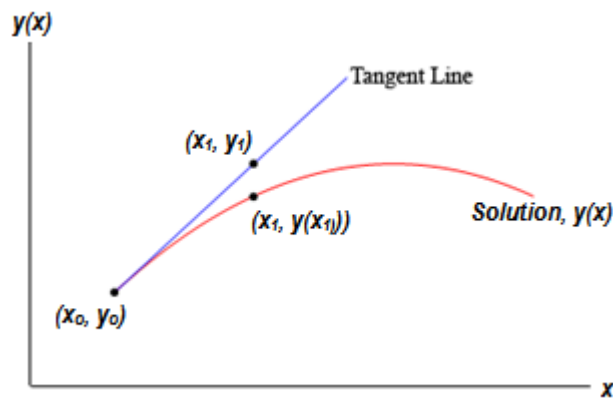
where  $f(x, y)$  is a known function and the values in the initial condition are also known numbers. We want to approximate the solution to (6) near  $x = x_0$ .

We'll start with the two pieces of information that we do know about the solution. First, we know the value of the solution at from the initial condition. Second, we also know the value of the derivative at . We can get this by plugging the initial condition  $y(x_0) = y_0$  into the differential equation itself. So, the derivative at this point is.

$$\left. \frac{dy}{dx} \right|_{x=x_0} = f(x_0, y_0)$$

Now, recall from Calculus that these two pieces of information are enough for us to write down the equation of the tangent line to the solution at . The tangent line is

$$y = y_0 + f(x_0, y_0) (x - x_0)$$



Take a look at the figure. If  $x_1$  is close enough to  $x_0$  then the point  $y_1$  on the tangent line should be fairly close to the actual value of the solution at  $x_1$ , or  $y(x_1)$ . Finding  $y_1$  is easy enough. All we need to do is plug  $x_1$  in the equation for the tangent line.

$$y_1 = y_0 + f(x_0, y_0) (x_1 - x_0) \quad (7)$$

Now, we would like to proceed in a similar manner, but we don't have the value of the solution at  $x_1$  and so we won't know the slope of the tangent line to the solution at this

point. This is a problem. We can partially solve it however, by recalling that  $y_1$  is an approximation to the solution at  $x_1$ . If  $y_1$  is a very good approximation to the actual value of the solution then we can use that to estimate the slope of the tangent line at  $x_1$ . So, let's hope that  $y_1$  is a good approximation to the solution and construct a line through the point  $(x_1, y_1)$  that has slope  $f(x_1, y_1)$ . This gives

$$y = y_1 + f(x_1, y_1) (x - x_1)$$

Now, to get an approximation to the solution at  $x = x_2$  we will hope that this new line will be fairly close to the actual solution at  $x_2$  and use the value of the line at  $x_2$  as an approximation to the actual solution. This gives.

$$y_2 = y_1 + f(x_1, y_1) (x_2 - x_1)$$

We can continue in this fashion. Use the previously computed approximation to get the next approximation. So

$$y_3 = y_2 + f(x_2, y_2) (x_3 - x_2)$$

$$y_4 = y_3 + f(x_3, y_3) (x_4 - x_3)$$

In general, if we have  $x_n$  and the approximation to the solution at this point,  $y_n$ , and we want to find the approximation at  $x_{n+1}$  all we need to do is use the following.

$$y_{n+1} = y_n + f(x_n, y_n) (x_{n+1} - x_n) \tag{8}$$

We can simplify the formula if we define

$$f_n = f(x_n, y_n)$$

and assume that

$$(x_{n+1} - x_n) = h$$

As often, we will assume that the step sizes between the points  $x_0, x_1, x_2, \dots$  are of a uniform size of  $h$ .

(It is important to note that this doesn't have to be done and there are times when it's best that we not do this.)

However, if we do the formula for the next approximation becomes.

$$y_{n+1} = y_n + h f_n \quad (9)$$

So, how do we use Euler's Method? It's fairly simple. We start with (6) and decide if we want to use a uniform step size or not. Then starting with  $(x_0, y_0)$  we repeatedly evaluate (8) or (9) depending on whether we chose to use a uniform step size or not. We continue until we've gone the desired number of steps or reached the desired time. This will give us a sequence of numbers  $y_1, y_2, y_3, \dots, y_n$  that will approximate the value of the actual solution at  $x_1, x_2, x_3, \dots, x_n$ .

An alternative method to obtain equation (9).

Beginning with equation (5)  $y' = f(x, y)$ ;  $x_0 \leq x \leq b$  with  $y(x_0) = y_0$  we write

$$dy = f(x, y) dx$$

Now integrate differential equation putting  $b = x_1$  between limits  $x_0$  and  $x_1$  as follows

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x, y) dx$$

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y) dx \quad (10)$$

If  $x_1$  is very close to  $x_0$  then  $y$  can be assumed as constant equal to  $y_0$  between limits  $x_0 < x < x_1$  and also replacing  $x$  by  $x_0$ , we can write

$$y(x_1) - y(x_0) = f(x_0, y_0) (x_1 - x_0)$$

Substituting 'h' for  $(x_1 - x_0)$  we obtain a relation identical to equation (7)

$$y_1 = y_0 + h f(x_0, y_0)$$

Thus moving step by step towards end point general solution can be written as equation (9)

$$y_{n+1} = y_n + h f_n.$$

In order to find acceptable solution  $h$  should be very small. To decide its value we go on repeating computations decreasing step size 'h' by half. When two consecutive results match with each other, computation is stopped.

**(2.2) Picard Iteration Method:**

*Picard method is an iterative method.* An iterative method gives a sequence of approximations  $y^{(1)}(x)$ ,  $y^{(2)}(x)$ , ...,  $y^{(k)}(x)$ , ... to the solution of differential equations such that the  $n$ th approximation is obtained from one or more previous approximations.

The integration of differential equation (10) yields

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y) dx$$

Note that the exact solution of IVP is obtained for  $x=x_0$  (initial condition)

For approximate solution, the exact solution  $y(x_1)$  is approximated by  $y^{(0)} = y_0$  in the integrand to get first iteration result

$$y^{(1)}(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x_0, y^{(0)}) dx$$

The approximation can be improved using general formula

$$y^{(n+1)}(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x_0, y^{(n)}) dx$$

From the results, it can be proved that the above sequence of approximations converges to the solution very near to the exact (closed form) solution of IVP.

**Prpblem** Obtain the approximate solution  $y(t)$  of IVP using Eulers method as well as Picatds method. Obtain approximate solution at  $x= 0.1$  correct to 4 places of decimal.

$$y' = 1+ xy, \quad y(0)=1$$

### Algorithm for Euler's Method

Input:

Step1-

- (1) Given differential equation in the form of a function as  $(dy/dx) = f(x, y)$
- (2) Initial value of function, that is,  $f(x_0) = y_0$ , where  $x_0$  is initial value of  $x$ .
- (3) End value of  $x$ , that is  $x_n$  and number of steps  $n$

Step 2 set  $x \leq x_0$  and  $y \leq y_0$

Step 3 calculate step size  $h \leq (x_n - x_0)/n$

Step 4 print  $x_0$ , “ ”,  $y_0$

Step 5 for  $i = 1$  to  $n$  do

$y_n = y_0 + h * f(x_0, y_0)$

$x_0 \leq x_0 + h$

$y_0 \leq y_n$

print  $x_0$ , “ ”,  $y_0$

end of  $i$  loop

step 6 stop

### Algorithm for Picard Method

Step1- inputs

- (1) Given differential equation in the form of a function as  $(dy/dx) = f(x, y)$
- (2) Initial value of function, that is,  $f(x_0) = y_0$ , where  $x_0$  is initial value of  $x$ .  
Read  $x_0$  and  $y_0$
- (3) End value of  $x$ , that is  $x_n$  and number of iterations  $n$
- (4) Read  $x_n$  and  $n$
- (5) Read allowed error  $a_{error}$

Step 2 set  $x \leq x_0$  and  $y_i \leq y_0$

Step 3 calculate step size  $h \leq (x_n - x_0)$

Step 4 print  $x_0$ , “ ”,  $y_0$

Step 5 for  $i = 1$  to  $n$  do

$y_n = y_i + h * f(x_0, y_0)$

print  $x_n$ , “ ”,  $y_n$

if  $abs(y_n - y_0) < a_{error}$  then stop

$y_0 \leq y_n$

end of  $i$  loop

print “ Maximum number of iterations reached”

step 6 stop