Lecture for Hermite Interpolation

HERMITE INTERPOLATION: An interpolation is said to be Hermite interpolation if it is involved both the function and its first derivative at each point of interpolation $((x_i, y_i, y'_i), i = 0, 1, 2, \cdots, n)$.

Therefore, it is satisfies the interpolation conditions at a given set of tabular points.

i.e. $p(x_i) = f(x_i), \; i = 0, 1, 2, \cdots, n$

and $p'(x_i) = f'(x_i), \; i = 0, 1, 2, \cdots, n.$

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∴ It is clear that, the Hermite interpolation polynomial $p(x)$ is satisfied $(2n+2)$ conditions and therefore the number of coefficients to be determined is $(2n+2)$.

Thus $p(x)$ must be a polynomial of degree $\leq (2n+1)$. Hence the required polynomial may be written as in the term of Lagrange interpolation in which involved both the function and its first derivative at a given set of tabular points

i.e. $$P_{2n+1}(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i) \qquad \text{——— (1)}$$

where $A_i(x)$ and $B_i(x)$ are polynomial of degree

$\leq 2n+1$ and satisfy

(i) $A_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

and $B_i(x_j) = 0 \ \forall \ i$ and j — (2)

(ii) $A_i'(x_j) = 0 \ \forall \ i$ and j — (3)

and $B_i'(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$\because$ we know that, in Lagrange fundamental polynomials we have $A_0(x) = \frac{x - x_1}{x_0 - x_1}$ and $A_1(x) = \frac{x - x_0}{x_1 - x_0}$

$\Rightarrow A_0(x_0) = \frac{x_0 - x_1}{x_0 - x_1}$ and $A_1(x_1) = \frac{x_1 - x_0}{x_1 - x_0}$

$\Rightarrow A_0(x_0) = 1$ and $A_1(x_1) = 1$

Similarly, $A_0(x_1) = 0$ and $A_1(x_0) = 0$

$\therefore$ we can generalize above result and show that $A_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Using the Lagrange fundamental polynomials $l_i(x)$, we write

$A_i(x) = u_i(x)[l_i(x)]^2$ — (4)

and $B_i(x) = v_i(x)[l_i(x)]^2$ — (5)

[where A_i & B_i are polynomial degree $\leq 2n+1$]

Since $l_i(x)$ is a polynomial of degree n

$\Rightarrow [l_i(x)]^2$ is a polynomial of degree $2n$

$\therefore$ It is clear that, $u_i(x)$ & $v_i(x)$ must be a linear polynomial.

Therefore, we can write

$$y_i(x) = a_i x + b_i$$

and $v_i(x) = c_i x + d_i$

Now, from (4) & (5), we have

$$\left.\begin{aligned} A_i(x) &= (a_i x + b_i)[l_i(x)]^2 \\ \text{and } B_i(x) &= (c_i x + d_i)[l_i(x)]^2 \end{aligned}\right] \quad (6)$$

Using the condition (2) in (6), we have

$$A_i(x_i) = (a_i x_i + b_i)[l_i(x_i)]^2$$

and $B_i(x_i) = (c x_i + d_i)[l_i(x_i)]^2$

$$\Rightarrow \quad 1 = (a_i x_i + b_i)[1]^2$$

and $0 = (c_i x_i + d_i)[1]^2$

$$\Rightarrow \quad a_i x_i + b_i = 1 \quad (7)$$

and $c_i x_i + d_i = 0 \quad (8)$

Again, using the condition (3) in (6), we have

$$A_i'(x_i) = 2(a_i x_i + b_i) l_i(x_i) l_i'(x_i) + a_i[l_i(x_i)]^2$$

$$0 = 2(1)(1) l_i'(x_i) + a_i[1]^2 \quad [\because a_i x_i + b_i = 1]$$

$$\Rightarrow \boxed{a_i = -2 l_i'(x_i)}$$

and $B_i'(x_i) = 2(c_i x_i + d_i) l_i(x_i) l_i'(x_i) + c_i[l_i(x_i)]^2$

$$1 = 2(0) l_i(x_i) l_i'(x_i) + c_i[1]^2 \quad [\because c_i x_i + d_i = 0]$$

$$\Rightarrow \boxed{c_i = 1} \quad \therefore \boxed{d_i = -x_i} \quad [\text{from } (8)]$$

Similarly, from (7)

$$b_i = 1 - a_i x_i$$

$$\Rightarrow \quad b_i = 1 + 2x_i \, l_i'(x_i)$$

Substituting the values of a_i, b_i, c_i and d_i in (6), we have $A_i(x)$ and $B_i(x)$.

Finally substituting the values of $A_i(x)$ and $B_i(x)$ in (1), we obtain

$$\boxed{P_{2n+1}(x) = \sum_{i=0}^{n} [1 - 2(x - x_i) \, l_i'(x_i)] \, l_i^2(x) f(x_i) + \sum_{i=0}^{n} (x - x_i) \, l_i^2(x) \, f'(x_i)}$$

which is the required Hermite Interpolating polynomial.

<u>Truncation Error:</u> The truncation error of Hermite interpolation polynomial is given by

$$E_{2n+1}(f, x) = \frac{w^2(x)}{(2n+2)!} f^{(2n+2)}(\xi), \; x_0 < \xi < x_n$$

where $w(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$