Number Field → A number field consists of a set F containing at least two distinct elements 0_F and 1_F, with two binary operations $+$ and $\times$, which are f^ns

$+ : F \times F \to F, \quad (r, s) \mapsto r + s \in F$

$\times : F \times F \to F, \quad (r, s) \mapsto r \times s \in F$

such that
(i) $(F, +)$ is an abelian group.
(ii) $(F, \times)$ is an abelian group.
(iii) it holds distributivity.

Eg:- (a) The set Q of all rational numbers, with ordinary addition and multiplication, forms a number field.
(b) The set R of all real numbers, with ordinary addition and multiplication, forms a number field.
(c) The set C of all complex numbers, with ordinary addition and multiplication, forms a number field.
(d) The set $Q[\sqrt{3}]$ of all combinations $p + q\sqrt{3}$ with rational numbers p and q, with the addition and multiplication of real numbers, is a number field, where for each non-zero $p + q\sqrt{3}$,

$$\frac{1}{p+q\sqrt{3}} = \frac{p}{[p^2 - 3q^2]} - \frac{q}{[p^2 - 3q^2]}\sqrt{3}$$

* Linear Space → A linear space over a field F consists of a set V containing at least one element 0_V, with two binary operations $\boxplus$ and

$\boxdot$, which are f^ns

$$\boxplus : V \times V \to V, \quad (u, w) \mapsto u \boxplus w \in V$$
$$\boxdot : F \times V \to V, \quad (r, v) \mapsto r \boxdot v \in V$$

such that

(i) $(V, \boxplus)$ is an abelian group

(ii) It holds associativity and multiplicative identity of $\boxdot$

(iii) It holds left and right distributivity.

Eg:- (a) Each number field F is a linear space over itself with $\boxplus = +_F$, $\boxdot = \times_F$ and $0_V = 0_F$.

(b) For each +ve integer n and each number field F, the set $F^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in F\}$ is a linear space over F, with "co-ordinatewise" operations :-

$$(x_1, x_2, \dots x_n) \boxplus (z_1, z_2, \dots, z_n) = (x_1 + z_1, \dots, x_n + z_n)$$
$$r \boxdot (x_1, x_2, \dots, x_n) = (r \cdot x_1, \dots, r \cdot x_n)$$
$$0_{F^n} = (0_F, \dots, 0_F)$$

★ **Linear Subspace** → A linear subspace of a linear space V over a field F consists of a subset $W \subseteq V$ that is also a linear space with the same element 0_V and with the same binary operations $\boxplus$ and $\boxdot$ already existing for V.

★ **Linear Map** → A linear function, also called Linear map, Linear mapping, Linear Operator, or Linear transformation, is a function $L : V \to W$ from a linear space V to a linear space W, with the same number field F for V and W such that

$$L(r.u + s.v) = r\cdot L(u) + s\cdot L(v)$$

for all $r, s \in F$ and all $u, v \in V$.

Eg:- The f^n $L: Q \to Q$ defined by $L(x) = 2x$ is linear because

$$L(r.u + s.v) = 2(ru + sv) = 2(ru) + 2(sv)$$
$$= r(2u) + s(2v) = r\cdot L(u) + s\cdot L(v)$$

$\forall\ r, s \in Q$ and all $u, v \in Q$

* Inner Product → An inner product (or a "scalar" or "dot" product) defined on a linear space V over a field $F \subseteq C$ is a function

$$\langle\ ,\ \rangle : V \times V \to F \qquad (u, w) \mapsto \langle u, w\rangle$$

which satisfies the following properties :-

(i) Non-negativity : $\langle v, v\rangle \geq 0_F$

(ii) Positivity : If $\langle v, v\rangle = 0_F$, then $v = 0_V$

(iii) Linearity : $\langle r \boxdot u \boxplus s \boxdot v, w\rangle = r\langle u, w\rangle +_F s\langle v, w\rangle$

(iv) Hermitian : $\langle u, w\rangle = \overline{\langle w, u\rangle}$

$\forall\ r, s \in F$ and $u, v, w \in V$

Eg:- (i) For each +ve integer n and each number field $F \subseteq C$, the set

$F^n = \{(x_1, x_2, ---, x_n) : x_1, x_2, ---, x_n \in F\}$ is a linear space over F, with inner product

$$\langle (x_1, x_2, ---, x_n), (z_1, z_2, ---, z_n)\rangle = x_1\overline{z_1} + x_2\overline{z_2} + --- + x_n\overline{z_n}$$

(ii) For each pair of real numbers $a < b$, the linear space $C^0([a, b], \mathbb{C})$ of all continuous f^n's $f: [a, b] \to \mathbb{C}$ with $f(t) = u(t) + iv(t)$, has an inner product defined by

$$\langle f, g\rangle = \int_a^b f(t)\,\overline{g(t)}\,dt$$