★ **Cover** → Let (X, T) be a topological space and let $A = \{A_\alpha : \alpha \in \Lambda\}$ be a collection of subsets of X.
If $\bigcup_{\alpha \in \Lambda} A_\alpha = X$, then A is called cover of X.

★ **Open Cover** → If elements of A are open sets, then A is called Open Cover of X.

★ **Finite Subcover** → If $\exists$ a finite subclass $A' = \{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ (say) of A such that $X = \bigcup_{i=1}^{n} A_{\alpha_i}$, then A' is called finite subcover of X.

(2013) (2011) (2002)

★ **Compact Space** → A topological space (X, T) is said to be compact if every open cover of X has finite subcover.

* Examples of Compact Spaces :-

① Every co-finite topological space is compact.
(2010)

PROOF → Let (X, T) be a co-finite topological space. Let $A = \{A_\alpha : \alpha \in \Delta\}$ be an open cover of X. Now, if A_{α_0} be any member of T, then $X - A_{\alpha_0}$ is finite.
So, let $X - A_{\alpha_0} = \{x_1, x_2, ----, x_n\}$.
$\because X - A_{\alpha_0} \subseteq X$ and A is an open cover of X, so each element of $X - A_{\alpha_0}$ is contained in one or the other member of A.
At the most, for each $x_i \in X - A_{\alpha_0}$, $\exists$ a set A_{α_i} in A such that $x_i \in A_{\alpha_i}$

$\therefore \quad X - A_{\alpha_0} \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}$

$\Rightarrow X = A_{\alpha_0} \cup (X - A_{\alpha_0}) \subseteq \bigcup_{i=0}^{n} A_{\alpha_i}$

This shows that $A' = \{A_{\alpha_i} : 0 \leq i \leq n\}$ is a finite subcover of A. Hence, (X, T) is compact.

(2011)
② Every finite set of topological space X is compact.

PROOF → Let $A = \{x_1, x_2, ----, x_n\}$ be a finite set.
Let $\{A_\alpha : \alpha \in \Delta\}$ be an open cover of A.
Then $x_1 \in A \Rightarrow x_1 \in A_{\alpha_1}$ for atleast one $\alpha_1 \in \Delta$
$x_2 \in A \Rightarrow x_2 \in A_{\alpha_2}$ for atleast one $\alpha_2 \in \Delta$
$x_3 \in A \Rightarrow x_3 \in A_{\alpha_3}$ for atleast one $\alpha_3 \in \Delta$
$\vdots$
$x_n \in A \Rightarrow x_n \in A_{\alpha_n}$ for atleast one $\alpha_n \in \Delta$

$\therefore \quad A = \{x_1, x_2, ----, x_n\} \subset A_{\alpha_1} \cup A_{\alpha_2} \cup ----- \cup A_{\alpha_n}$

or

$$A \subset \bigcup_{i=1}^{n} A_{\alpha_i}$$

This shows that open cover of A has a finite subcover. Hence, A is compact.

★ **Sequential Compactness** → Let X be a topological space. If $\langle x_n \rangle$ is a sequence of points of X and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of +ve integers, then the sequence $\langle y_i \rangle$ defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence $\langle x_n \rangle$. The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

★ **Countable Compactness** → A set E in a topological space (X, T) is said to be countably compact iff every countable open cover of E has a finite sub-cover.

(2013) (2010)

• **Theorem-16** → Every sequentially compact topological space is countably compact.

PROOF → Let (X, T) be a sequentially compact topological space. If X is finite, then it is clearly countably compact. So, let X be infinite and let if possible, X be not countably compact. Then $\exists$ a countable open covering

$A = \{A_n : n \in N\}$ of X, which does not have a finite sub-cover.

Let $x_1 \in X \cap A_{n_1}$. Let n_2 be the least +ve integer greater than n_1 such that $X \cap A_{n_2} \neq \phi$. Let $x_2 \in (X \cap A_{n_2}) - (X \cap A_{n_1})$. Such a point will surely exist, for, otherwise A_{n_1} will cover X. Continuing this process, we obtain a sequence $\langle x_n \rangle$ of points of X such that,

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for each $i \in N$,
$$x_i \in X \cap A_{n_i} \text{ and } x_i \notin \bigcup_{K=1}^{(i-1)} (X \cap A_{n_K})$$
We claim that $\langle x_n \rangle$ has no convergent subsequence in X.
Let $x \in X$. Then $\exists$ an open set A_m in A such that $x \in A_m$. $\because X \cap A_m \neq \phi$, $\therefore \exists\, K \in N$ such that $A_m = A_{n_K}$.
But, by our choice of $\langle x_n \rangle$, we have
$$i > K \Rightarrow x_i \notin A_{n_K}.$$
Thus A_{n_K} is an open set containing x and it does not contain all the points of the sequence. Consequently, no subsequence of $\langle x_n \rangle$ converges to x. So, it follows that X is not sequentially compact, which is a contradiction.
Hence, (X, T) is countably compact.
(2013)
• Theorem-2 → Let Y be a subspace of a topological space X and let $A \subset Y$. Then A is compact relative to X iff A is compact relative to Y.

PROOF → Let (X,T) be a topological space and $A \subset Y \subset X$.
First of all, we assume that A is compact relative to X, then we have to prove that A is compact relative to Y.
Let $\{A'_\alpha : \alpha \in \Delta\}$ be an open cover of A relative to Y, i.e., $A \subset \bigcup_{\alpha \in \Delta} A'_\alpha$
$\because A'_\alpha \in T_Y$, $\therefore \exists\, A_\alpha \in T$ such that
$$A'_\alpha = A_\alpha \cap Y \in T_Y$$
Now,
$$A \subset \bigcup_{\alpha \in \Delta} A'_\alpha$$
$$\Rightarrow A \subset \bigcup_{\alpha \in \Delta} [A_\alpha \cap Y]$$
$$\Rightarrow A \subset Y \cap [\bigcup_{\alpha \in \Delta} A_\alpha]$$
$$\Rightarrow A \subset \bigcup_{\alpha \in \Delta} A_\alpha$$

This shows that $\{A_\alpha : \alpha \in \Delta\}$ becomes an open cover of A in X. But A is compact relative to X, $\therefore \exists$ a finite subcover $\{A_{\alpha_1}, A_{\alpha_2}, ----, A_{\alpha_n}\}$ (say) of A in X, i.e.,

$$A \subset \bigcup_{i=1}^{n} A_{\alpha_i}$$

Now, $A \subset Y$ and $A \subset \bigcup_{i=1}^{n} A_{\alpha_i}$

$\Rightarrow A \subset Y \cap \left[\bigcup_{i=1}^{n} A_{\alpha_i}\right]$

$\Rightarrow A \subset \bigcup_{i=1}^{n} \left[A_{\alpha_i} \cap Y\right]$

$\Rightarrow A \subset \bigcup_{i=1}^{n} A'_{\alpha_i}$

This shows that every open cover of A in Y has a finite subcover. $\therefore$ A is compact relative to Y.

Conversely, we assume that A is compact relative to Y, then we have to prove that A is compact relative to X.

Let $\{A_\alpha : \alpha \in \Delta\}$ be an open cover of A in X, i.e.,

$$A \subset \bigcup_{\alpha \in \Delta} A_\alpha$$

Also, $A \subset Y$, $\therefore A \subset Y \cap \left[\bigcup_{\alpha \in \Delta} A_\alpha\right]$

$\Rightarrow A \subset \bigcup_{\alpha \in \Delta} \left[A_\alpha \cap Y\right]$

$\Rightarrow A \subset \bigcup_{\alpha \in \Delta} A'_\alpha$ where $A'_\alpha = A_\alpha \cap Y \in T_Y$

This shows that $\{A'_\alpha : \alpha \in \Delta\}$ is an open cover of A in Y. But A is compact in Y, $\therefore \exists$ a finite subcover $\{A'_{\alpha_1}, A'_{\alpha_2}, ----, A'_{\alpha_n}\}$ (say) of A in Y, i.e.,

$$A \subset \bigcup_{i=1}^{n} A'_{\alpha_i}$$

$\Rightarrow A \subset \bigcup_{i=1}^{n} \left[A_{\alpha_i} \cap Y\right]$

$\Rightarrow A \subset Y \cap \left[\bigcup_{i=1}^{n} A_{\alpha_i}\right]$

$\Rightarrow A \subset \bigcup_{i=1}^{n} A_{\alpha_i}$

This shows that $\{A_{\alpha_1}, A_{\alpha_2}, ----, A_{\alpha_n}\}$ is a finite subcover of A in X. Thus A is compact relative to X.

Hence, proved.