Gauss – Legendre Integration Methods

Introduction:

The general problem of numerical integration is to find an approximate value of the integral

$$\boxed{I = \int_a^b w(x) f(x)\,dx} \quad \text{--- (1).}$$

where, $w(x) > 0$ in $[a, b]$ is the weight function. We assume that $w(x)$ and $w(x)f(x)$ are integrable in the Riemann sense on $[a, b]$. The limits of integration may be finite, semi-finite or infinite.

The integral (1) is approximated by a finite linear combination of values of $f(x)$ in the form

$$\boxed{I = \int_a^b w(x) f(x)\,dx \simeq \sum_{k=0}^{n} \lambda_k f_k} \quad \text{--- (2)}$$

where, x_k, $k = 0, 1, \ldots n$ are called the nodes distributed within the limits of integration $[a, b]$ and λ_k, $k = 0, 1, \ldots n$ are called the weights of the integration rule.

Newton - Cotes Methods

When $w(x)=1$ and the nodes x_k's are equispaced with $x_0 = a$, $x_n = b$ with spacing $h = \frac{b-a}{n}$, the methods given in equation (2) are called Newton - Cotes integration methods. The weights λ_k's are called Cotes numbers.

Gauss Quadrature Methods:

In the integration method given in eqn (2), the Nodes x_k's and the weights λ_k's $k=0,1,\dots n$, can also be obtained by making the formula exact for polynomials of degree upto m. when the nodes are known, that is $m=n$, the corresponding methods are called Newton-Codes methods. When the nodes are also to be determined, we have $m=2n+1$, and the methods are called Gaussian integration methods.

Since any finite interval $[a,b]$ can always be transformed to $[-1,1]$, using the transformation

$$\boxed{x = \frac{b-a}{2}t + \frac{b+a}{2}}$$

we consider the integral in the form

$$\boxed{\int_{-1}^{1} w(x) f(x)\,dx = \sum_{k=0}^{n} \lambda_k f_k} \quad \text{———} (3)$$

where, $w(x) > 0$, $-1 \le x \le 1$, is the weight function.

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1. Gauss – Legendre Integration Methods.

Taking the weight function $\boxed{w(x) = 1}$, the method given in equation 3 reduces to

$$\boxed{\int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{n} \lambda_k f(x_k)} \quad \text{—— (1.1)}$$

One-point formula: $n = 0$, the formula is given by

$$\boxed{\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0)} \quad \text{—— (1.2)}$$

The method has two unknowns λ_0 and x_0. Making the method exact for $f(x) = 1, x$, we get,

If $f(x) = 1$, then from eq^n (1.2), we have,

$$\int_{-1}^{1} 1\,dx = \lambda_0 \cdot 1. \qquad [\because f(x) = 1 \;\forall x.$$

$$\Rightarrow [x]_{-1}^{1} = \lambda_0.$$

$$\Rightarrow \boxed{2 = \lambda_0}$$

$$\therefore \boxed{f(x) = 1 : \; 2 = \lambda_0}.$$

If $f(x) = x$, then from eq^n (1.2), we have,

$$\int_{-1}^{1} x\,dx = \lambda_0 \cdot x_0 \qquad \left[\because f(x) = x \;\forall x. \;\Rightarrow f(x_0) = x_0.\right.$$

$$\Rightarrow \left[\frac{x^2}{2}\right]_{-1}^{1} = \lambda_0 x_0.$$

$$\Rightarrow 0 = \lambda_0 x_0 \text{ or } x_0 = 0.$$

$$\boxed{f(x) = x \quad : \quad x_0 = 0}$$

Hence, substituting these values of λ_0 and x_0 in eq^n (1·2), we get,

$$\boxed{\int_{-1}^{1} f(x)\,dx = 2f(0).}$$

The error constant is given by.

$$C = \int_{-1}^{1} x^2\,dx - 2[0] \qquad \left[\because \begin{array}{l} f(x) = x^2 \\ f(0) = 0 \end{array}\right.$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

Hence, the error associated in this method is given by.

$$R_1 = \frac{C}{2!} f''(\xi)$$

$$= \frac{1}{2} \times \frac{2}{3} f''(\xi) \qquad -1 < \xi < 1.$$

$$\therefore \boxed{R_1 = \frac{1}{3} f''(\xi).}, \quad -1 < \xi < 1.$$

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Two - point formula : $n=1$:

The formula is given by.

$$\boxed{\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)} \quad \text{—— (1.3)}.$$

The method has four unknowns, x_0, x_1, λ_0 and λ_1. Making the method exact for $f(x) = 1, x, x^2, x^3$, we get,

Taking $f(x) = 1$, we get,

$$\int_{-1}^{1} 1\,dx = \lambda_0 + \lambda_1$$

$$\Rightarrow 2 = \lambda_0 + \lambda_1$$

$$\therefore \boxed{f(x) = 1 : \; 2 = \lambda_0 + \lambda_1} \quad \text{—— (1.3a)}$$

Taking $f(x) = x$, we get.

$$\int_{-1}^{1} x\,dx = \lambda_0 x_0 + \lambda_1 x_1$$

$$\Rightarrow 0 = \lambda_0 x_0 + \lambda_1 x_1.$$

$$\therefore \boxed{f(x) = x : \; 0 = \lambda_0 x_0 + \lambda_1 x_1} \quad \text{—— (1.3b)}.$$

Taking $f(x) = x^2$, we get,

$$\int_{-1}^{1} x^2\,dx = \lambda_0 x_0^2 + \lambda_1 x_1^2$$

$$\Rightarrow \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2.$$

[$\because f(x) = x^2$.
$\therefore f(x_0) = x_0^2$
& $f(x_1) = x_1^2$.]

$\therefore \boxed{f(x) = x^2 : \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2}$ — (1·3c)

Taking $f(x) = x^3$, we get.

$$\int_{-1}^{1} x^3 dx = \lambda_0 x_0^3 + \lambda_1 x_1^3.$$

$$\Rightarrow 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3.$$

$\therefore \boxed{f(x) = x^3 : 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3}$ — (1·3d).

From (1·3 b) and (1·3 d), multiplying (1·3 b) by x_0^2 on both sides and subtracting from (1·3d)

we get,

$$0 = \lambda_0 x_0^3 + \lambda_1 x_1^3$$

$$(-) \quad 0 = \lambda_0 x_0^3 + \lambda_1 x_1 x_0^2$$

$$0 = \lambda_1 x_1 (x_1^2 - x_0^2).$$

$$\Rightarrow \lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0 \qquad [\because a^2 - b^2 = (a+b)(a-b).$$

Since, $\lambda_1 \neq 0$, $x_0 \neq x_1$, we get, $x_1 + x_0 = 0$ i.e. $\boxed{x_1 = -x_0}$.

and $x_1 \neq 0$ since $x_0 \neq x_1$ and $\lambda_1 \neq 0$.

Substituting in (1·3b) we get,

$$\lambda_0 x_0 - \lambda_1 x_0 = 0. \qquad [\because x_1 = -x_0.$$

$$\Rightarrow x_0(\lambda_0 - \lambda_1) = 0.$$

$$\Rightarrow \lambda_0 - \lambda_1 = 0. \qquad [\because x_0 \neq 0.$$

$$\Rightarrow \boxed{\lambda_0 = \lambda_1}$$

Substituting in (1·3a), we get,

$$2 = \lambda_0 + \lambda_0$$

$$\Rightarrow 2\lambda_0 = 2.$$

$$\Rightarrow \boxed{\lambda_0 = 1}$$

and since $\lambda_0 = \lambda_1$

$$\therefore \boxed{\lambda_0 = \lambda_1 = 1}$$

Using (1·3c), we get,

$$\frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2$$

$$\Rightarrow \frac{2}{3} = x_0^2 + x_1^2 \quad [\because \lambda_0 = \lambda_1 = 1.$$

$$\Rightarrow \frac{2}{3} = x_0^2 + (-x_0)^2 \quad [\because x_1 = -x_0.$$

$$\Rightarrow \frac{2}{3} = x_0^2 + x_0^2.$$

$$\Rightarrow \frac{2}{3} = 2x_0^2.$$

$$\Rightarrow \frac{1}{3} = x_0^2.$$

$$\Rightarrow \boxed{x_0 = \pm\frac{1}{\sqrt{3}}}.$$

and we ~~know that~~ already have,

$$x_1 = -x_0.$$

$$\Rightarrow x_1 = -\left(\pm\frac{1}{\sqrt{3}}\right).$$

$$\Rightarrow \boxed{x_1 = \mp\frac{1}{\sqrt{3}}}$$

Therefore, the two-point Gauss-Legendre method is given by (using equation 1.3).

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1).$$

$$\therefore \int_{-1}^{1} f(x)\,dx = 1\cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1\cdot f\left(\frac{1}{\sqrt{3}}\right).$$

$$\Rightarrow \boxed{\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

$\because \lambda_0 = \lambda_1 = 1$
$x_0 = \pm \frac{1}{\sqrt{3}}$
$x_1 = \mp \frac{1}{\sqrt{3}}$.

$\because f(x) = x^4$

The error constant is given by.

$$C = \int_{-1}^{1} x^4\,dx - \left[\left(-\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4\right].$$

$$= \left[\frac{x^5}{5}\right]_{-1}^{1} - \left[\frac{1}{9} + \frac{1}{9}\right].$$

$$= \frac{2}{5} - \frac{2}{9} = \frac{18-10}{45} = \frac{8}{45}.$$

The associated error in this formula is given by.

$$R_4 = \frac{C}{4!} f^{(4)}(\xi) = \frac{8}{45} \times \frac{1}{2\times3\times4} f^{(4)}(\xi).$$

$$\Rightarrow \boxed{R_4 = \frac{1}{135} f^{(4)}(\xi)}, \quad -1 < \xi < 1.$$

To do: Derive three-point formula following the same procedure.