M. Sc. (Computer Science) 2nd Sem
MCS 201

(2) Gauss - Chebyshev Integration Methods:

Taking the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ in the equation

$$\boxed{\int_{-1}^{1} w(x) f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k} \quad \text{——— } (2.1)$$

where, $w(x) > 0$, $-1 \le x \le 1$, is the weight function,

We have,

$$\boxed{\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx = \sum_{k=0}^{n} \lambda_k f(x_k)} \quad \text{——— } (2.2)$$

The nodes x_k and weights λ_k are unknown.

Let us consider the following cases,

(i) One - point formula: $n = 0$, the formula given in equation (2.2) becomes.

$$\boxed{\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx = \lambda_0 f(x_0)} \quad \text{——— } (2.3)$$

This method has two unknowns λ_0, x_0.

Making it exact for $f(x) = 1, x$, we get from (2.3),

If $f(x) = 1$; then $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lambda_0$. $\left[\because f(x) = 1 \ \forall x. \ f(x_0) = 1.\right.$

or $\left[\sin^{-1} x\right]_{-1}^{1} = \lambda_0$ $\left[\because \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x.\right.$

$\Rightarrow$ $\boxed{\pi = \lambda_0}$ $\left[\because \sin^{-1}(1) = \frac{\pi}{2}.\right.$

$\therefore$ $\boxed{f(x) = 1 \ ; \ \lambda_0 = \pi}$

If $f(x) = x$, then

$$\int_{-1}^{1} \frac{x\,dx}{\sqrt{1-x^2}} = \lambda_0 x_0$$

[$\because f(x) = x \;\forall x$
$\therefore f(x_0) = x_0$.]

$$\Rightarrow 0 = \lambda_0 x_0 .$$

$$\Rightarrow \boxed{x_0 = 0} \quad \cancel{[\lambda_0 \neq 0]}.$$

[Hint: For the integration of $\frac{x}{\sqrt{1-x^2}}$,
use substitution method
taking $x = \sin t$ (say).
then $dx = \cos t\,dt$.]

Hence, the method is given by

$$\boxed{\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \pi f(0)}$$

The error constant is given by

$$C = \int_{-1}^{1} \frac{x^2\,dx}{\sqrt{1-x^2}} - \pi.0 .$$

[taking $f(x) = x^2$
Since we make the method is exact for $1, x$.]

$$= \int_{-1}^{1} \frac{x^2\,dx}{\sqrt{1-x^2}}$$

$$= 2\int_{0}^{1} \frac{x^2\,dx}{\sqrt{1-x^2}} = 2\int_{0}^{\pi/2} \sin^2\theta\,d\theta .$$

$$= \frac{\pi}{2} .$$

[Use the substitution
$x = \sin\theta$.
as $x \to 0,\ \theta \to 0$
as $x \to 1,\ \theta \to \pi/2$.]

$\therefore$ The error in this method is

$$\boxed{R_1 = \frac{C}{2!} f''(\xi) = \frac{\pi}{4} f''(\xi)}, \quad -1 < \xi < 1.$$

Proceeding in the same manner, we get,

(ii) Two-point formula:

$$\boxed{\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)\right]}.$$

and the error term R_4 is given by.

$$R_4 = \frac{C}{4!} f^{(4)}(\eta) \quad \text{and} \quad C = \frac{\pi}{8}. \quad , -1<\eta<1,$$

$$\therefore \boxed{R_4 = \frac{\pi}{192} f^{(4)}(\eta)} \quad -1<\eta<1.$$

(iii) Three-point formula:

$$\boxed{\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]}.$$

and the error term R_6 is given by

$$R_6 = \frac{C}{6!} f^{(6)}(\eta)$$ where the error constant

$$C = \frac{\pi}{32}$$

$$\therefore \boxed{R_6 = \frac{\pi}{23040} f^{(6)}(\eta)} \quad -1<\eta<1.$$

To do: Evaluate the integral

$$I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx .$$

using the Gauss-Chebyshev 1-point, 2-point and 3-point quadrature rules. Evaluate it also using the Gauss-Legendre 3-point formula.