M.Sc. (Computer Science) 2nd Sem ①
MCS 201

Gauss-Legendre Integration Methods

① **One-point formula:**

$$\boxed{\int_{-1}^{1} f(x)\,dx = 2f(0)}$$

and the ~~associated~~ error, $\boxed{R_1 = \frac{1}{3} f''(\xi)}$, $-1<\xi<1$

② **Two-point formula:**

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

and the error, $\boxed{R_4 = \frac{1}{135} f^{(4)}(\xi)}$, $-1<\xi<1$

③ **Three-point formula:**

$$\boxed{\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]}$$

and the error, $\boxed{R_6 = \frac{1}{15750} f^{(6)}(\xi)}$, $-1<\xi<1$

[* $f^{(n)}$ denotes n^{th} derivative of f]

M.Sc. (computer Science) 2nd Sem

MCS 201

Examples on Gauss-Legendre Integration method :-

Example: Evaluate the integral $I = \int_1^2 \frac{2x\,dx}{1+x^4}$ using the Gauss-Legendre 1-point, 2-point and 3-point methods. Compare the results with the exact solution $I = \tan^{-1}(4) - (\pi/4)$.

Solution:- To use the Gauss-Legendre method, first ~~we~~ the given interval [1, 2] is to be reduced to [−1, 1]. For this we use the transformation

$$x = \frac{b-a}{2}t + \frac{b+a}{2}.$$

Here, $a = 1$, $b = 2$.

$$\therefore x = \frac{2-1}{2}t + \frac{2+1}{2}.$$

$$\Rightarrow x = \frac{1}{2}t + \frac{3}{2}.$$

$$\therefore \boxed{x = \frac{t+3}{2}} \text{ and } \boxed{dx = \frac{1}{2}dt}.$$

Using these results in the given integral we have,

$$I = \int_{-1}^{1} \frac{2 \cdot \frac{t+3}{2}}{1+\left(\frac{t+3}{2}\right)^4} \cdot \frac{1}{2}dt.$$

$$\Rightarrow \boxed{I = \int_{-1}^{1} \frac{8(t+3)}{16+(t+3)^4}dt = \int_{-1}^{1} f(t)\,dt} \text{ (Say)}$$

where, $f(t) = \frac{8(t+3)}{16+(t+3)^4}$.

(i) **1-point formula,**

We know that the Gauss-Legendre 1-point rule is given by.

$$\boxed{\int_{-1}^{1} f(x)\,dx = 2f(0)}$$

$$\therefore\ I = 2f(0) = 2\left[\frac{8(0+3)}{16+(0+3)^4}\right] \qquad \left[\because f(t) = \frac{8(t+3)}{16+(t+3)^4}\right.$$

$$= 2\left(\frac{24}{97}\right)$$

$$\therefore\ \boxed{I \simeq 0.4948.}$$

(ii) **Using 2-point formula :**

Gauss-Legendre 2-point formula is given by

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

$$\therefore\ I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{8\left(-\frac{1}{\sqrt{3}}+3\right)}{16+\left(-\frac{1}{\sqrt{3}}+3\right)^4} + \frac{8\left(\frac{1}{\sqrt{3}}+3\right)}{16+\left(\frac{1}{\sqrt{3}}+3\right)^4}$$

$$= \frac{8(-0.5774+3)}{16+(-0.5774+3)^4} + \frac{8(0.5774+3)}{16+(0.5774+3)^4}$$

$$= 0.3842 + 0.1592$$

$$\therefore\ \boxed{I = 0.5434}$$

(iii) 3 - point formula

Using Gauss - Legendre 3 - point formula, we have,

$$\boxed{\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\tfrac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\tfrac{3}{5}}\right)\right]}$$

$$\therefore I = \frac{1}{9}\left[5 \cdot \frac{8\left(-\sqrt{\frac{3}{5}}+3\right)}{16+\left(-\sqrt{\frac{3}{5}}+3\right)^4} + 8\left[\frac{8(0+3)}{16+(0+3)^4}\right] + 5 \cdot \frac{8\left(\sqrt{\frac{3}{5}}+3\right)}{16+\left(\sqrt{\frac{3}{5}}+3\right)^4}\right]$$

$$= \frac{1}{9}\left[5\left(\frac{17.8032}{40.5263}\right) + 8(0.2474) + 5\left(\frac{30.1968}{218.9943}\right)\right]$$

$$= \frac{1}{9}\left[5(0.4393) + 8(0.2474) + 5(0.1379)\right]$$

$$= \frac{1}{9}(4.8652)$$

$$\therefore \boxed{I = \frac{4.8652}{9} = 0.5406}$$

The exact solution is.

$$I = \tan^{-1}(4) - (\pi/4)$$

$$= \cancel{75.9638}$$

$$= 1.3258 - 0.7854$$

$$\boxed{I = 0.5404}$$

Comparing all the above results we can see that Gauss Legendre 3-point method gives the solution with more accuracy.

To do: Evaluate the integrals

(i) $I = \int_0^2 \frac{dx}{3+4x}$. (ii) $\int_0^2 \frac{dx}{x^2+2x+10}$

(iii) $I = \int_0^1 \frac{dx}{1+x}$.

by Gauss - Legendre two - point and three - point formulas.