(1) Removable singularity :→
An isolated singularity of $f(z)$ at $z = a$ is said to be a removable singularity if $\lim_{z \to a} f(z)$ exist then $f(z)$ can be redefined at $z = a$ so that it become analytic at $z = a$

Ex :- $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$. bec. if we define

$$f(z) = \frac{\sin z}{z} \quad \text{if } z \neq 0$$

$$= 1 \quad \text{if } z = 1$$

then $f(z)$ is analytic at $z = 0$

(2) Pole :→ An isolated singularity $z = a$ of $f(z)$ is said to be a pole for if $\lim_{z \to a} f(z) = \infty$

In this case $f(z)$ is defined if z is near a and $z \neq a$

Eg :- $f(z) = \frac{1}{z^n}$ $(n > 0$ and $n \in N)$

In this case $\lim_{z \to 0} f(z) = \infty$ and

$$\lim_{z \to 0} z^n f(z) = 1$$

$z = 0$ is pole (or n^{th} order) for f

③ Essential Singularity → An isolated singularity $z = a$ of $f(z)$ is said to be essential singularity of f if $\lim_{z \to 0} f(z)$ does not exist

Thus ~~an~~ essential sing. is neither removable sing. or pole.

$$f(z) = e^{1/z}$$

Theo → Riemann's Theo. on removable sing. →

State → If f has an isolated singularity at z_0 then $z = z_0$ is a removable sing. iff one of the following condition holds

(I) f is bounded in a ~~closed~~ deleted nbhd of z_0.

(II) $\lim_{z \to z_0} f(z)$ exist.

(III) $\lim_{z \to z_0} (z - z_0) f(z) = 0$

proof: we suppose that f has isolated sing at z_0. then $f(z)$ is analytic in a deleted nbhd of z_0 i.e. $\exists \delta > 0$ such that if $0 < |z - z_0| < \delta$ then ~~such~~ $f(z)$ is analytic

first we assume that one of the given condition holds.

we assume that condition (ii) hold

i.e.

$$\lim_{z \to z_0} (z - z_0) f(z) = 0$$

Now, we introduce a function $h(z)$ as follows

$$h(z) = \begin{cases} (z - z_0) f(z) & \text{if } 0 < |z - z_0| < \delta \\ 0 & \text{if } z = z_0 \end{cases}$$

clearly $h(z)$ consider with $f(z)$ in the region $0 < |z - z_0| < \delta$ hence $h(z)$ is also analytic in this region.

further h is continuous at $z = z_0$

because by def.

$$\lim_{z \to z_0} h(z) = \lim_{z \to z_0} (z - z_0) f(z) = 0 = h(z_0)$$

$$\lim_{z \to z_0} h(z) = h(z_0)$$

Thus h is analytic at $z = z_0$ and throughout the region $|z - z_0| < \delta$

(cor 4 of)

Now we define another function

$$g(z) = \frac{h(z) - h(z_0)}{z - z_0} = \frac{h(z)}{(z - z_0)}$$

$(\because h(z_0) = 0)$

Clearly $\lim_{z \to z_0} g(z)$ exists and equal a $h'(z$

also $f = g$ for $z \neq z_0$ (by def)

$\left(h(z) = (z - z_0) f(z) \quad \frac{h(z)}{z - z_0} = f(z) = g(z) \right)$

Hence we define $f(z_0)$ as $f(z_0) = h'(z_0)$

Thus $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z)$

$= h'(z_0)$

$= f(z_0)$

$h(z) = (z-z_0) f(z)$

$h'(z) = (z-z_0) f'(z) + f(z)$

$h'(z_0) = f(z_0)$

Thus f is continuous at z_0 and since analytic in the region $|z-z_0| < \delta$

i.e. The singularity $z = z_0$ of f has been removed.

$z = z_0$ is a removable singularity for f.

If Cond. (I) hold.

then clearly (III) also hold

because if f is bounded then

$\lim_{z \to z_0} (z-z_0) f(z) = 0$

Similarly (II) $\Rightarrow$ (III) because if $\lim_{z \to z_0} f(z)$ exist then $\lim_{z \to z_0} (z-z_0) f(z) = 0$

thus $z = z_0$ is removable sing. if any one of the condition (I) (II) (III) holds.

Next we assume that f has a removable sing. at $z = z_0$. Then by def.

$\lim_{z \to z_0} f(z)$ exist

hence it is obvious that $f(z)$ is bounded in a deleted neigh. of z.

(i.e in the region $0 < |z - z_0| < \delta$)

and also $\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} f(z)$

$= 0$

thus (I) (II) (III) holds.

Thus complete the proof.

<u>Remark</u> → Let f and g be analytic and suppose both have a zero of order n at $z = z_0$. Thus we can write.

$f(z) = (z - z_0)^n f_0(z)$

$g(z) = (z - z_0)^n g_0(z)$

where f_0 & g_0 are analytic and non-zero at $z = z_0$

let $h(z) = \dfrac{f(z)}{g(z)}$ $z \neq z_0$

clearly

$$\lim_{z \to z_0} h(z) = \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

which exist

Hence $h(z)$ has a removable sing. at $z = z_0$

Ex →

let $f(z) = e^z - 1$ and $g(z) = z$

the $z = 0$ is a zero of both $f(z)$ and

$\dfrac{f(z)}{g(z)} = \dfrac{e^z - 1}{z}$ has a removable sing. at $z =$

POLE :- we know that an isolated singu. $z = z_0$ of f is a pole if there exist a positive integer n such that $(z-z_0)^n f(z)$ is bounded near z_0

(Equivalently $z = z_0$ is a pole if
$$\lim_{z \to z_0} f(z) = \infty$$

or if $\lim_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$)

The smallest integer n such that $(z-z_0)^n f(z)$ is bounded is called the order of the pole at $z = z_0$

If $n = 1$, we say that z_0 is a simple Pole.

Example :- $$f(z) = \frac{z \cos(\pi z/2a)}{(z-a)(z^2+b^2)^7 \sin^5 z}$$

where a, b are distinct nonzero real no.

Solve :- Then $f(z)$ has a pole of order 7 a $z = \pm ib$, a pole of order ~~5~~ at $z = \pm K(\ldots$
A pole of order 4 at $z = 0$ and removable sing. at $z = a$

because
$$(z^2+b^2)^7 f(z) = \frac{(z^2+b^2)^7 z \cos(z\pi/2a)}{(z-a)(z^2+b^2)^7 \sin^5 z}$$

$$= \frac{z \cos(z\pi/2a)}{(z-a)\sin^5 z}$$

is bounded near $z = \pm ib$

$$z^4 f(z) = \frac{z^5 \cos(\pi z/2a)}{(z-a)(z^2+b^2)^7 \sin^5 z}$$

$$= \left(\frac{z}{\sin z}\right)^5 \cdot \frac{\cos(\pi z/2a)}{(z-a)(z^2+b^2)^7}$$

is bounded near $z = 0$

Similarly for $z = \pm k\pi \quad (k \in \mathbb{N})$

$$(z \pm k\pi)^5 f(z) = \frac{(z \pm k\pi)^5 z \cos(z\pi/2a)}{(z-a)(z^2+b^2)^7 \sin(z \pm k\pi \mp k\pi)}$$

$$= \frac{(z \pm k\pi)^5 \cdot z \cos(z\pi/2a)}{(z-a)(z^2+b^2)^7 [\sin(z \pm k\pi)\cos(\mp k\pi) - \sin(k\pi)\cos(z \pm k\pi)]}$$

$$= \left(\frac{(z \pm k\pi)}{\sin(z \pm k\pi)}\right)^5 \frac{z \cos(z\pi/2a)}{(-1)^{\pm k\pi} (z-a)(z^2+b^2)^7}$$

which is bounded near $z = \pm k\pi$

$f(z)$ has a pole of order 5 at $z = \pm k\pi$

finaly, near $z = a$

$$(z-a) f(z) = \frac{(z-a) z \cos(\pi z/2a)}{(z-a)(z^2+b^2)^7 \sin^5 z} = \frac{z \cos(\pi z/2a)}{(z^2+b^2)^7 \cdot \sin^5 z}$$

$(z-a)f(z) \longrightarrow 0$ as $z \longrightarrow a$ $\left(\cos(\pi z/2a) \to 0\right)$

$\lim (z-a) f(z) = 0$

Therefore $f(z)$ has a removable sing. at $z = a$

Theo → (1.12) If f is analytic in a deleted nbd. of z_0, then f has a pole at z_0 iff there exist an n such that $(z-z_0)^n f(z)$ is bounded near z_0.

more precisely,

f has a pole of order n iff $\lim_{z \to z_0} (z-z_0)^n f(z) \neq 0$ and $(z-z_0)^n f(z)$ has a removable singularity at z_0

i.e $\lim_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$

proof → If f is analytic in a deleted nbd of z_0 i.e z_0 is an isolated sing. of f.

Now

By def, z_0 is pole of f of order n iff $(z-z_0)^n f(z)$ is bounded.

Now $\phi(z) = (z-z_0)^n f(z)$

by def of removable sing.

Then $\phi(z)$ is bounded near z_0

$\Leftrightarrow$ $\phi(z)$ has a removable sing. at z_0

$\Leftrightarrow$ $\lim_{z \to z_0} \phi(z)$ exist and is non zero.

$\Rightarrow \lim_{z \to z_0} (z-z_0)\, Q(z) = 0$

Also in this case,

$$\lim f(z) = \lim \frac{Q(z)}{(z-z_0)^n} = \infty$$

Thus $f(z)$ has a pole of order n at z_0 iff any above condition holds

This prove the theorem.

Theo 7.14 → Let f be analytic in a deleted nbhd of z_0. Then function f has a pole of order n at $z = z_0$ iff.

There are positive constant c_1 & c_2 such that

$$c_2 < |(z-z_0)^n f(z)| \le c_1$$

for some deleted nbhd N_0 of z_0 such that $N_0 \subset N$.

proof →

first we assume that the given inequalities hold.

i.e

$$c_2 \le |(z-z_0)^n f(z)| \le c_1$$

from right inequality, we note that

$(z-z_0)^n f(z)$ is bounded
hence $(z-z_0)^{n+1} f(z) \to 0$ as $z \to z_0$

Also from left ineq.
$|(z-z_0)^n f(z)| \geq C_2$
as $z \to z_0$
(i.e $\lim (z-z_0)^n f(z) \to 0$)
hence $z = z_0$ is a pole of order n.

Conversely, let f have a pole of order n at $z = z_0$
Then by def, $(z-z_0)^n f(z)$ is bounded near

if we put $g(z) = (z-z_0)^n f(z)$
Then $g(z)$ is bounded near z_0. also $g(z)$ ha[illegible] a removable sing. at z_0
hence $\lim_{z \to z_0} g(z)$ exist and is not zero
$\exists$ positive constant C_1 & C_2
$C_2 < |g(z)| < C_1$
$C_2 < |(z-z_0)^n f(z)| < C_1$
this prove the theo.